# Lecture 1

## Tools for Analysis of Nonlinear Systems

Anton Shiriaev
Stockholm, 2018-09-03

**Learning outcomes:** Concepts of stability of a motion. Stable cycles of nonlinear systems. Tools for analysis – Lyapunov lemma, Poincare first return map, small parameter and Krylov-Bogolyubov methods for approximate integration. Examples.

 

# Outline

# Orbital Stability vs. Stability of a Motion

# Orbital Stability vs. Stability of a Motion

Consider a nonlinear dynamic system and one of its solutions

$$\tfrac{d}{dt}x = f(x), \quad x^*(t) = x^*(t, x_0^*) \in \mathbb{R}^n, \ t \in [0, +\infty)$$

**Definition (Lyapunov Stability)**

The solution $x^*(\cdot)$ is called

- stable, if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that,

$$\text{if} \quad \|x(0) - x_0^*\| < \delta, \quad \text{then} \quad \|x(t) - x^*(t)\| < \varepsilon \quad \forall\, t \geq 0$$

- asymptotically stable, if, in addition,

$$\lim_{t \to +\infty} \|x(t) - x^*(t)\| = 0.$$

# Orbital Stability vs. Stability of a Motion

Consider a nonlinear system and one of its periodic solutions

$$\frac{d}{dt}x = f(x), \quad x^*(t) = x^*(t, x_0^*), \quad x^*(t) = x^*(t+T), \ \forall\, t, \ \ T > 0$$

Let $\Gamma_{x^*} \subset \mathbb{R}^n$ denote the orbit of $x^*(\cdot)$

$$\Gamma_{x^*} = \left\{\xi \in \mathbb{R}^n : \ \xi = x^*(t), \ t \in [0, T]\right\}.$$

**Definition (Orbital Stability)**

The periodic solution $x^*(\cdot)$ is called

- orbitally stable, if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that,

$$\text{if} \quad \|x(0) - x_0^*\| < \delta, \quad \text{then} \quad \operatorname{dist}\left(x(t), \Gamma_{x^*}\right) < \varepsilon \quad \forall\, t \geq 0$$

- asymptotically orbitally stable, if, in addition,

$$\lim_{t\to+\infty} \operatorname{dist}\left(x(t), \Gamma_{x^*}\right) = 0$$

Unfortunately there is no one general arguments
to answer the questions:

When does a dynamical system have a periodic solution?
How to find it by analyzing its model?

If present for the model, when can it be observed in experiment,
that is, when is the periodic motion stable?

Important Observation:

Periodic solutions are usually encountered in 2nd order systems,
and rarely observed for systems of higher dimensions!

Unfortunately there is no one general arguments
to answer the questions:

When does a dynamical system have a periodic solution?
How to find it by analyzing its model?

If present for the model, when can it be observed in experiment,
that is, when is the periodic motion stable?

Important Observation:

Periodic solutions are usually encountered in 2nd order systems,
and rarely observed for systems of higher dimensions!

# Preliminary observations

Unfortunately there is no one general arguments
to answer the questions:

When does a dynamical system have a periodic solution?
How to find it by analyzing its model?

If present for the model, when can it be observed in experiment,
that is, when is the periodic motion stable?

Important Observation:

Periodic solutions are usually encountered in 2nd order systems,
and rarely observed for systems of higher dimensions!

# Search for an Equilibrium vs. Detecting a Cycle

To find an equilibrium $x_0$ of the dynamical system

$$\frac{d}{dt}x = f(x)$$

one needs only to solve algebraic equation

$$f(x_0) = 0$$

---

To find a $T$-periodic solution $x(t) = x(t + T)$ of the system

$$\frac{d}{dt}x = f(x)$$

one needs, in general, to solve/integrate this system

$\Uparrow$

Except particular cases, it is impossible!

## Search for an Equilibrium vs. Detecting a Cycle

To find an equilibrium $x_0$ of the dynamical system

$$\frac{d}{dt}x = f(x)$$

one needs only to solve algebraic equation

$$f(x_0) = 0$$

---

To find a $T$-periodic solution $x(t) = x(t+T)$ of the system

$$\frac{d}{dt}x = f(x)$$

one needs, in general, to solve/integrate this system

$\Uparrow$

Except particular cases, it is impossible!

# Lyapunov's theorems

# Lyapunov theorems. Preliminaries:

Consider a 2nd order system

$$\begin{aligned}\dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y)\end{aligned}$$

Suppose that $(x_0, y_0)$ is its equilibrium, i.e.

$$P(x_0, y_0) = Q(x_0, y_0) = 0$$

Introduce the linearization of the system at $(x_0, y_0)$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

If

$$P'_x(x_0, y_0)Q'_y(x_0, y_0) - P'_y(x_0, y_0)Q'_x(x_0, y_0) \neq 0$$

the equilibrium $(x_0, y_0)$ is called simple.

## Lyapunov theorems. Preliminaries:

Consider a 2nd order system

$$\begin{aligned}\dot{x} &= P(x, y)\\ \dot{y} &= Q(x, y)\end{aligned}$$

Suppose that $(x_0, y_0)$ is its equilibrium, i.e.

$$P(x_0, y_0) = Q(x_0, y_0) = 0$$

Introduce the linearization of the system at $(x_0, y_0)$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

If

$$P'_x(x_0, y_0) Q'_y(x_0, y_0) - P'_y(x_0, y_0) Q'_x(x_0, y_0) \neq 0$$

the equilibrium $(x_0, y_0)$ is called simple.

# Lyapunov theorems. Preliminaries:

Consider a 2nd order system

$$\dot{x} = P(x, y)$$
$$\dot{y} = Q(x, y)$$

Suppose that $(x_0, y_0)$ is its equilibrium, i.e.

$$P(x_0, y_0) = Q(x_0, y_0) = 0$$

Introduce the linearization of the system at $(x_0, y_0)$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0) \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

If

$$P'_x(x_0, y_0)Q'_y(x_0, y_0) - P'_y(x_0, y_0)Q'_x(x_0, y_0) \neq 0$$

the equilibrium $(x_0, y_0)$ is called simple.

## Lyapunov theorems. Preliminaries:

Consider a 2nd order system

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned}$$

Suppose that $(x_0, y_0)$ is its equilibrium, i.e.

$$P(x_0, y_0) = Q(x_0, y_0) = 0$$

Introduce the linearization of the system at $(x_0, y_0)$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

If

$$P'_x(x_0, y_0) Q'_y(x_0, y_0) - P'_y(x_0, y_0) Q'_x(x_0, y_0) \neq 0$$

the equilibrium $(x_0, y_0)$ is called simple.

# Lyapunov theorems. Preliminaries (Cont'd):

Consider the case when the linearized system is

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad b > 0$$

It is easy to check that:

All solutions are periodic

---

All solutions are of the same period $T = \frac{2\pi}{b}$

---

The function

$$z_1^2(t) + z_2^2(t) = C = z_1^2(0) + z_2^2(0)$$

is the integral of the system

 

## Lyapunov theorems. Preliminaries (Cont'd):

Consider the case when the linearized system is

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad b > 0$$

It is easy to check that:

All solutions are periodic

---

All solutions are of the same period $T = \frac{2\pi}{b}$

---

The function

$$z_1^2(t) + z_2^2(t) = C = z_1^2(0) + z_2^2(0)$$

is the integral of the system

## Lyapunov theorems. Preliminaries (Cont'd):

Consider the case when the linearized system is

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad b > 0$$

It is easy to check that:

All solutions are periodic

---

All solutions are of the same period $T = \frac{2\pi}{b}$

---

The function

$$z_1^2(t) + z_2^2(t) = C = z_1^2(0) + z_2^2(0)$$

is the integral of the system

Consider the case when the linearized system is

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad b > 0$$

It is easy to check that:

All solutions are periodic

---

All solutions are of the same period $T = \frac{2\pi}{b}$

---

The function

$$z_1^2(t) + z_2^2(t) = C = z_1^2(0) + z_2^2(0)$$

is the integral of the system

## Lyapunov theorems. Preliminaries (Cont'd):

The linearization is a cheap computational procedure

---

Question: Is it true that if the linearization

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

has a periodic solution, then the original nonlinear system

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned}$$

has one too?

---

Answer: Unfortunately, NO!

The linearization is a cheap computational procedure

---

Question: Is it true that if the linearization

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

has a periodic solution, then the original nonlinear system

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned}$$

has one too?

---

Answer: Unfortunately, NO!

## Lyapunov theorems. Preliminaries (Cont'd):

The linearization is a cheap computational procedure

---

Question: Is it true that if the linearization

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

has a periodic solution, then the original nonlinear system

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned}$$

has one too?

---

Answer: Unfortunately, NO!

## Example:

Consider the system

$$\begin{aligned} \dot{x} &= -y - x(x^2 + y^2) \\ \dot{y} &= x - y(x^2 + y^2) \end{aligned}$$

---

Its linearization at the equilibrium $x = y = 0$

$$\begin{aligned} \dot{z}_1 &= -z_2 \\ \dot{z}_2 &= z_1 \end{aligned}$$

has the centre at $z_1 = z_2 = 0$!

---

Unfortunately, this nonlinear system has

NO periodic motion at all !!!

## Example:

Consider the system

$$\dot{x} = -y - x(x^2 + y^2)$$
$$\dot{y} = x - y(x^2 + y^2)$$

---

Its linearization at the equilibrium $x = y = 0$

$$\dot{z}_1 = -z_2$$
$$\dot{z}_2 = z_1$$

has the centre at $z_1 = z_2 = 0$!

---

Unfortunately, this nonlinear system has

NO periodic motion at all !!!

## Example (Cont'd):

Indeed, in the polar coordinates

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

the differential equation for radius $\rho$ is decoupled

$$\begin{cases} \dot{x} = -y - x(x^2 + y^2) \\ \dot{y} = x - y(x^2 + y^2) \end{cases} \quad \Rightarrow \quad \dot{\rho} = -\rho^3$$

The solution of the last equation is

$$\rho(t) = \frac{\rho(0)}{\sqrt{2t\cdot\rho^2(0)+1}} \to 0 \quad \text{as} \quad t \to +\infty$$

## Example (Cont'd):

Indeed, in the polar coordinates

$$\begin{cases} x &= \rho\cos\theta \\ y &= \rho\sin\theta \end{cases}$$

the differential equation for radius $\rho$ is decoupled

$$\begin{cases} \dot{x} &= -y - x(x^2+y^2) \\ \dot{y} &= x - y(x^2+y^2) \end{cases} \quad \Rightarrow \quad \dot{\rho} = -\rho^3$$

---

The solution of the last equation is

$$\rho(t) = \frac{\rho(0)}{\sqrt{2t\cdot\rho^2(0)+1}} \ \to \ 0 \quad \text{as} \quad t \to +\infty$$

## Conclusions from the Example:

- Existence of a periodic solution for the linearized system

  DOES NOT IMPLY

  the presence of a cycle in the nonlinear original system

- Establishing a presence of a periodic solution of a nonlinear system by linearization is

  NONTRIVIAL TASK

## Conclusions from the Example:

- Existence of a periodic solution for the linearized system

  DOES NOT IMPLY

  the presence of a cycle in the nonlinear original system

- Establishing a presence of a periodic solution of a nonlinear system by linearization is

  NONTRIVIAL TASK

# Lyapunov Theorems. Preliminaries (Cont'd):

Consider the system

$$\begin{aligned} \dot{x} &= -by + \phi(x, y) \\ \dot{y} &= bx + \psi(x, y) \end{aligned}$$

Here $b > 0$ and the functions $\phi$, $\psi$ are analytical

$$\begin{aligned} \phi(x, y) &= P_2(x, y) + P_3(x, y) + \cdots + P_k(x, y) + \ldots \\ \psi(x, y) &= Q_2(x, y) + Q_3(x, y) + \cdots + Q_k(x, y) + \ldots \end{aligned}$$

where $P_k(x, y)$, $Q_k(x, y)$ are homogeneous polynomials of degree $k$

$$\begin{aligned} P_2(x, y) &= p_{21}\, x^2 + p_{22}\, xy + p_{23}\, y^2 \\ P_3(x, y) &= p_{31}\, x^3 + p_{32}\, x^2 y + p_{33}\, xy^2 + p_{34}\, y^3 \\ &\vdots \end{aligned}$$

# Lyapunov Theorems. Preliminaries (Cont'd):

Consider the system

$$\begin{aligned} \dot{x} &= -by + \phi(x, y) \\ \dot{y} &= bx + \psi(x, y) \end{aligned}$$

Here $b > 0$ and the functions $\phi$, $\psi$ are analytical

$$\begin{aligned} \phi(x, y) &= P_2(x, y) + P_3(x, y) + \cdots + P_k(x, y) + \ldots \\ \psi(x, y) &= Q_2(x, y) + Q_3(x, y) + \cdots + Q_k(x, y) + \ldots \end{aligned}$$

where $P_k(x, y)$, $Q_k(x, y)$ are homogeneous polynomials of degree $k$

$$\begin{aligned} P_2(x, y) &= p_{21}\, x^2 + p_{22}\, xy + p_{23}\, y^2 \\ P_3(x, y) &= p_{31}\, x^3 + p_{32}\, x^2 y + p_{33}\, xy^2 + p_{34}\, y^3 \\ &\vdots \end{aligned}$$

# Lyapunov Theorems. Preliminaries (Cont'd):

In the polar coordinates

$$x = \rho\cos\theta, \quad y = \rho\sin\theta$$

the dynamics

$$\begin{cases} \dot{x} &= -by + \phi(x,y) \\ \dot{y} &= bx + \psi(x,y) \end{cases}$$

is

$$\begin{cases} \dot{x} &= \dot{\rho}\cos\theta - \rho\sin\theta\dot{\theta} &= -b\rho\sin\theta + \phi(\rho\cos\theta, \rho\sin\theta) \\ \dot{y} &= \dot{\rho}\sin\theta + \rho\cos\theta\dot{\theta} &= b\rho\cos\theta + \psi(\rho\cos\theta, \rho\sin\theta) \end{cases}$$

Therefore

$$\begin{cases} \dot{\rho} = \dot{x}\cos\theta + \dot{y}\sin\theta = \phi(\rho\cos\theta, \rho\sin\theta)\cos\theta + \psi(\rho\cos\theta, \rho\sin\theta)\sin\theta \\ \dot{\theta} = \dfrac{\dot{y}\cos\theta - \dot{x}\sin\theta}{\rho} = b + \dfrac{\psi(\rho\cos\theta, \rho\sin\theta)\cos\theta - \phi(\rho\cos\theta, \rho\sin\theta)\sin\theta}{\rho} \end{cases}$$

# Lyapunov Theorems. Preliminaries (Cont'd):

In the polar coordinates

$$x = \rho\cos\theta, \quad y = \rho\sin\theta$$

the dynamics

$$\begin{cases} \dot{x} &= -by + \phi(x,y) \\ \dot{y} &= bx + \psi(x,y) \end{cases}$$

is

$$\begin{cases} \dot{x} &= \dot{\rho}\cos\theta - \rho\sin\theta\dot{\theta} &= -b\rho\sin\theta + \phi(\rho\cos\theta, \rho\sin\theta) \\ \dot{y} &= \dot{\rho}\sin\theta + \rho\cos\theta\dot{\theta} &= b\rho\cos\theta + \psi(\rho\cos\theta, \rho\sin\theta) \end{cases}$$

Therefore

$$\begin{cases} \dot{\rho} = \dot{x}\cos\theta + \dot{y}\sin\theta = \phi(\rho\cos\theta,\rho\sin\theta)\cos\theta + \psi(\rho\cos\theta,\rho\sin\theta)\sin\theta \\ \\ \dot{\theta} = \dfrac{\dot{y}\cos\theta - \dot{x}\sin\theta}{\rho} = b + \dfrac{\psi(\rho\cos\theta,\rho\sin\theta)\cos\theta - \phi(\rho\cos\theta,\rho\sin\theta)\sin\theta}{\rho} \end{cases}$$

# Lyapunov Theorems. Preliminaries (Cont'd):

The equations

$$\begin{aligned}\dot\rho &= \phi(\rho\cos\theta, \rho\sin\theta)\cos\theta + \psi(\rho\cos\theta, \rho\sin\theta)\sin\theta \\ \dot\theta &= b + \frac{\psi(\rho\cos\theta, \rho\sin\theta)\cos\theta - \phi(\rho\cos\theta, \rho\sin\theta)\sin\theta}{\rho}\end{aligned}$$

can be rewritten as

$$\begin{aligned}\dot\rho &= \underbrace{\Big[P_2(x,y)+P_3(x,y)+\dots\Big]}_{\phi(x,y)}\cos\theta + \underbrace{\Big[Q_2(x,y)+Q_3(x,y)+\dots\Big]}_{\psi(x,y)}\sin\theta \\ &= \Big[P_2(\rho\cos\theta,\rho\sin\theta)+\dots\Big]\cos\theta+\Big[Q_2(\rho\cos\theta,\rho\sin\theta)+\dots\Big]\sin\theta \\ &= \rho^2\Big[P_2(\cos\theta,\sin\theta)\cos\theta + Q_2(\cos\theta,\sin\theta)\sin\theta\Big] + \dots \\ \dot\theta &= b + \rho\Big[Q_2(\cos\theta,\sin\theta)\cos\theta - P_2(\cos\theta,\sin\theta)\sin\theta\Big] + \dots\end{aligned}$$

---

Equilibrium $\rho = 0$ is simple (i.e. $b > 0$ ) $\Rightarrow \dot\theta \neq 0$ for small $\rho$

The equations

$$\begin{aligned} \dot{\rho} &= \phi(\rho\cos\theta, \rho\sin\theta)\cos\theta + \psi(\rho\cos\theta, \rho\sin\theta)\sin\theta \\ \dot{\theta} &= b + \frac{\psi(\rho\cos\theta, \rho\sin\theta)\cos\theta - \phi(\rho\cos\theta, \rho\sin\theta)\sin\theta}{\rho} \end{aligned}$$

can be rewritten as

$$\begin{aligned} \dot{\rho} &= \underbrace{\Big[P_2(x,y) + P_3(x,y) + \ldots\Big]}_{\phi(x,y)}\cos\theta + \underbrace{\Big[Q_2(x,y) + Q_3(x,y) + \ldots\Big]}_{\psi(x,y)}\sin\theta \\ &= \Big[P_2(\rho\cos\theta, \rho\sin\theta) + \ldots\Big]\cos\theta + \Big[Q_2(\rho\cos\theta, \rho\sin\theta) + \ldots\Big]\sin\theta \\ &= \rho^2\Big[P_2(\cos\theta, \sin\theta)\cos\theta + Q_2(\cos\theta, \sin\theta)\sin\theta\Big] + \ldots \\ \dot{\theta} &= b + \rho\Big[Q_2(\cos\theta, \sin\theta)\cos\theta - P_2(\cos\theta, \sin\theta)\sin\theta\Big] + \ldots \end{aligned}$$

Equilibrium $\rho = 0$ is simple (i.e. $b > 0$ ) $\Rightarrow \dot{\theta} \neq 0$ for small $\rho$

# Lyapunov Theorems. Preliminaries (Cont'd):

The equations

$$\begin{aligned}\dot{\rho} &= \phi(\rho\cos\theta, \rho\sin\theta)\cos\theta + \psi(\rho\cos\theta, \rho\sin\theta)\sin\theta \\ \dot{\theta} &= b + \frac{\psi(\rho\cos\theta, \rho\sin\theta)\cos\theta - \phi(\rho\cos\theta, \rho\sin\theta)\sin\theta}{\rho}\end{aligned}$$

can be rewritten as

$$\begin{aligned}\dot{\rho} &= \underbrace{\Big[P_2(x,y)+P_3(x,y)+\dots\Big]}_{\phi(x,y)}\cos\theta + \underbrace{\Big[Q_2(x,y)+Q_3(x,y)+\dots\Big]}_{\psi(x,y)}\sin\theta \\ &= \Big[P_2(\rho\cos\theta,\rho\sin\theta)+\dots\Big]\cos\theta+\Big[Q_2(\rho\cos\theta,\rho\sin\theta)+\dots\Big]\sin\theta \\ &= \rho^2\Big[P_2(\cos\theta,\sin\theta)\cos\theta + Q_2(\cos\theta,\sin\theta)\sin\theta\Big] + \dots \\ \dot{\theta} &= b + \rho\Big[Q_2(\cos\theta,\sin\theta)\cos\theta - P_2(\cos\theta,\sin\theta)\sin\theta\Big] + \dots\end{aligned}$$

---

Equilibrium $\rho = 0$ is simple (i.e. $b > 0$ ) $\Rightarrow \dot{\theta} \neq 0$ for small $\rho$

# Lyapunov Theorems. Preliminaries (Cont'd):

Let us exclude time $t$ from consideration

$$\frac{d\rho}{d\theta} = \frac{\rho^2\left[P_2(\cos\theta,\sin\theta)\cos\theta + Q_2(\cos\theta,\sin\theta)\sin\theta\right] + \dots}{b + \rho\left[Q_2(\cos\theta,\sin\theta)\cos\theta - P_2(\cos\theta,\sin\theta)\sin\theta\right] + \dots} = R(\rho,\theta)$$

---

The function in the right-hand side is

- periodic in $\theta$: $R(\rho,\theta) = R(\rho, 2\pi+\theta)$
- for any $\theta$: $R(\rho,\theta)\Big|_{\rho=0} = 0$
- analytic function in some vicinity of $\rho = 0$

$$R(\rho,\theta) = \rho R_1(\theta) + \rho^2 R_2(\theta) + \rho^3 R_3(\theta) + \dots$$

# Lyapunov Theorems. Preliminaries (Cont'd):

Let us exclude time $t$ from consideration

$$\frac{d\rho}{d\theta} = \frac{\rho^2 \left[P_2(\cos\theta, \sin\theta)\cos\theta + Q_2(\cos\theta, \sin\theta)\sin\theta\right] + \ldots}{b + \rho\left[Q_2(\cos\theta, \sin\theta)\cos\theta - P_2(\cos\theta, \sin\theta)\sin\theta\right] + \ldots} = R(\rho, \theta)$$

---

The function in the right-hand side is

- periodic in $\theta$: $R(\rho, \theta) = R(\rho, 2\pi + \theta)$
- for any $\theta$: $R(\rho, \theta)\Big|_{\rho=0} = 0$
- analytic function in some vicinity of $\rho = 0$

$$R(\rho, \theta) = \rho R_1(\theta) + \rho^2 R_2(\theta) + \rho^3 R_3(\theta) + \ldots$$

Let us exclude time $t$ from consideration

$$\frac{d\rho}{d\theta} = \frac{\rho^2\left[P_2(\cos\theta,\sin\theta)\cos\theta + Q_2(\cos\theta,\sin\theta)\sin\theta\right]+\ldots}{b+\rho\left[Q_2(\cos\theta,\sin\theta)\cos\theta - P_2(\cos\theta,\sin\theta)\sin\theta\right]+\ldots} = R(\rho,\theta)$$

---

The function in the right-hand side is

- periodic in $\theta$: $R(\rho,\theta) = R(\rho, 2\pi+\theta)$
- for any $\theta$: $R(\rho,\theta)\Big|_{\rho=0} = 0$
- analytic function in some vicinity of $\rho = 0$

$$R(\rho,\theta) = \rho R_1(\theta) + \rho^2 R_2(\theta) + \rho^3 R_3(\theta) + \ldots$$

# Lyapunov Theorems. Preliminaries (Cont'd):

Let us exclude time $t$ from consideration

$$\frac{d\rho}{d\theta} = \frac{\rho^2\left[P_2(\cos\theta,\sin\theta)\cos\theta + Q_2(\cos\theta,\sin\theta)\sin\theta\right] + \dots}{b + \rho\left[Q_2(\cos\theta,\sin\theta)\cos\theta - P_2(\cos\theta,\sin\theta)\sin\theta\right] + \dots} = R(\rho,\theta)$$

---

The function in the right-hand side is

- periodic in $\theta$: $R(\rho,\theta) = R(\rho, 2\pi + \theta)$
- for any $\theta$: $R(\rho,\theta)\Big|_{\rho=0} = 0$
- analytic function in some vicinity of $\rho = 0$

$$R(\rho,\theta) = \rho R_1(\theta) + \rho^2 R_2(\theta) + \rho^3 R_3(\theta) + \dots$$

# Lyapunov Theorems. Preliminaries (Cont'd):

Denote

$$\rho(\theta) = f\left(\theta; \theta_0, \rho_0\right)$$

the solution of the differential equation

$$\tfrac{d}{d\theta}\rho = R(\rho, \theta)$$

with the origin in $[\theta_0, \rho_0]$. Then

- $f\left(\theta; \theta_0, \rho_0\right)\Big|_{\rho_0=0} = 0;$
- $f$ is the analytic function

$$f\left(\theta; \theta_0, \rho_0\right) = u_1(\theta)\rho_0 + u_2(\theta)\rho_0^2 + \cdots + u_k(\theta)\rho_0^k + \ldots$$

- all the weighting functions in the expansion

$$u_1(\theta), \quad u_2(\theta), \quad \ldots, \quad u_k(\theta), \quad \ldots$$

can be recursively found.

# Lyapunov Theorems. Preliminaries (Cont'd):

Denote

$$\rho(\theta) = f\left(\theta; \theta_0, \rho_0\right)$$

the solution of the differential equation

$$\tfrac{d}{d\theta}\rho = R(\rho, \theta)$$

with the origin in $[\theta_0, \rho_0]$. Then

- $f\left(\theta; \theta_0, \rho_0\right)\Big|_{\rho_0=0} = 0$;
- $f$ is the analytic function

$$f\left(\theta; \theta_0, \rho_0\right) = u_1(\theta)\rho_0 + u_2(\theta)\rho_0^2 + \cdots + u_k(\theta)\rho_0^k + \dots$$

- all the weighting functions in the expansion

$$u_1(\theta), \quad u_2(\theta), \quad \dots, \quad u_k(\theta), \quad \dots$$

can be recursively found.

# Lyapunov Theorems. Preliminaries (Cont'd):

Denote

$$\rho(\theta) = f\left(\theta; \theta_0, \rho_0\right)$$

the solution of the differential equation

$$\tfrac{d}{d\theta}\rho = R(\rho, \theta)$$

with the origin in $[\theta_0, \rho_0]$. Then

- $f\left(\theta; \theta_0, \rho_0\right)\Big|_{\rho_0=0} = 0$;
- $f$ is the analytic function

$$f\left(\theta; \theta_0, \rho_0\right) = u_1(\theta)\rho_0 + u_2(\theta)\rho_0^2 + \cdots + u_k(\theta)\rho_0^k + \ldots$$

- all the weighting functions in the expansion

$$u_1(\theta), \quad u_2(\theta), \quad \ldots, \quad u_k(\theta), \quad \ldots$$

can be recursively found.

# Lyapunov Theorems. Preliminaries (Cont'd):

Denote

$$\rho(\theta) = f\left(\theta; \theta_0, \rho_0\right)$$

the solution of the differential equation

$$\tfrac{d}{d\theta}\rho = R(\rho, \theta)$$

with the origin in $[\theta_0, \rho_0]$. Then

- $f\left(\theta; \theta_0, \rho_0\right)\Big|_{\rho_0=0} = 0;$
- $f$ is the analytic function

$$f\left(\theta; \theta_0, \rho_0\right) = u_1(\theta)\rho_0 + u_2(\theta)\rho_0^2 + \cdots + u_k(\theta)\rho_0^k + \ldots$$

- all the weighting functions in the expansion

$$u_1(\theta), \quad u_2(\theta), \quad \ldots, \quad u_k(\theta), \quad \ldots$$

  can be recursively found.

# First Lyapunov Theorem

The function

$$f\left(\theta;\theta_0,\rho_0\right)\Big|_{\theta=2\pi,\,\theta_0=0} = u_1(2\pi)\rho_0 + u_2(2\pi)\rho_0^2 + \dots$$
$$= \alpha_1\rho_0 + \alpha_2\rho_0^2 + \dots + \alpha_k\rho_0^k + \dots$$

is the Poincare first return map; constants $\{\alpha_1, \alpha_2, \dots\}$ are the focus quantities.

---

Theorem: *The first nonzero coefficient of the series*

$$f\left(2\pi;0,\rho_0\right) - \rho_0 = (\alpha_1 - 1)\rho_0 + \alpha_2\rho_0^2 + \dots + \alpha_k\rho_0^k + \dots$$

*has an odd number.* ■

---

Q: When does the system have the center at its equilibrium?

A: If and only if $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \dots = 0$

# First Lyapunov Theorem

The function

$$f\left(\theta;\theta_0,\rho_0\right)\Big|_{\theta=2\pi,\,\theta_0=0} = u_1(2\pi)\rho_0 + u_2(2\pi)\rho_0^2 + \dots$$
$$= \alpha_1\rho_0 + \alpha_2\rho_0^2 + \dots + \alpha_k\rho_0^k + \dots$$

is the Poincare first return map; constants $\{\alpha_1, \alpha_2, \dots\}$ are the focus quantities.

---

Theorem: *The first nonzero coefficient of the series*

$$f\left(2\pi;0,\rho_0\right) - \rho_0 = (\alpha_1 - 1)\rho_0 + \alpha_2\rho_0^2 + \dots + \alpha_k\rho_0^k + \dots$$

*has an odd number.* ■

---

Q: When does the system have the center at its equilibrium?

A: If and only if $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \dots = 0$

# First Lyapunov Theorem

The function

$$f\left(\theta;\theta_0,\rho_0\right)\Big|_{\theta=2\pi,\,\theta_0=0} = u_1(2\pi)\rho_0 + u_2(2\pi)\rho_0^2 + \dots$$
$$= \alpha_1\rho_0 + \alpha_2\rho_0^2 + \dots + \alpha_k\rho_0^k + \dots$$

is the Poincare first return map; constants $\{\alpha_1, \alpha_2, \dots\}$ are the focus quantities.

---

Theorem: *The first nonzero coefficient of the series*

$$f\left(2\pi;0,\rho_0\right) - \rho_0 = (\alpha_1 - 1)\rho_0 + \alpha_2\rho_0^2 + \dots + \alpha_k\rho_0^k + \dots$$

*has an odd number.* ■

---

Q: When does the system have the center at its equilibrium?

A: If and only if $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \dots = 0$

# Second Lyapunov Theorem:

*The necessary and sufficient conditions for the system*

$$\begin{aligned} \dot{x} &= -by + \phi(x, y) \\ \dot{y} &= bx + \psi(x, y) \end{aligned}$$

*to have the center at its equilibrium* $(x_0, y_0)$ *provided that* $\phi(x, y)$, $\psi(x, y)$ *are analytic functions, are that in some neighborhood of* $(x_0, y_0)$ *the system has an integral* $H(x, y)$, *which is an analytic function. Furthermore, it has the form*

$$x^2 + y^2 + \Phi_3(x, y) + \Phi_4(x, y) + \cdots + \Phi_k(x, y) + \cdots = C$$

*where* $\Phi_k(x, y)$ *are homogeneous polynomials of degree* $k$. ■

---

Important: If $\phi(x, y)$, $\psi(x, y)$ are not analytic function, the statement is not true!

## Second Lyapunov Theorem:

*The necessary and sufficient conditions for the system*

$$\begin{aligned} \dot{x} &= -by + \phi(x, y) \\ \dot{y} &= bx + \psi(x, y) \end{aligned}$$

*to have the center at its equilibrium* $(x_0, y_0)$ *provided that* $\phi(x, y)$, $\psi(x, y)$ *are analytic functions, are that in some neighborhood of* $(x_0, y_0)$ *the system has an integral* $H(x, y)$, *which is an analytic function. Furthermore, it has the form*

$$x^2 + y^2 + \Phi_3(x, y) + \Phi_4(x, y) + \cdots + \Phi_k(x, y) + \cdots = C$$

*where* $\Phi_k(x, y)$ *are homogeneous polynomials of degree* $k$. ■

---

Important: If $\phi(x, y)$, $\psi(x, y)$ are not analytic function, the statement is not true!

## Example:

The dynamics of a point-mass pendulum is

$$\ddot{\theta} + a \cdot \sin\theta = 0$$

Its linearization around the downward equilibrium $\theta = 0$ is

$$\ddot{z} + a \cdot z = 0$$

The pendulum has an energy as the integral

$$E(\theta, \dot{\theta}) = \dot{\theta}^2 + 2a\,(1 - \cos\theta) = \dot{\theta}^2 + a \cdot \theta^2 + \ldots$$

$\Rightarrow$ Pendulum has the center around its downward position $\Leftarrow$

---

What is about the system

$$\ddot{\theta} + \left[\sin\left(100|\theta|\right) - \sqrt{|\theta|}\right]\dot{\theta}^2 + a \cdot \sin\theta = 0 \quad ???$$

It has the center too, see the following Lectures.

# Example:

The dynamics of a point-mass pendulum is

$$\ddot{\theta} + a \cdot \sin\theta = 0$$

Its linearization around the downward equilibrium $\theta = 0$ is

$$\ddot{z} + a \cdot z = 0$$

The pendulum has an energy as the integral

$$E(\theta, \dot{\theta}) = \dot{\theta}^2 + 2a\,(1 - \cos\theta) = \dot{\theta}^2 + a \cdot \theta^2 + \ldots$$

$\Rightarrow$ Pendulum has the center around its downward position $\Leftarrow$

---

What is about the system

$$\ddot{\theta} + \left[\sin\left(100|\theta|\right) - \sqrt{|\theta|}\right]\dot{\theta}^2 + a \cdot \sin\theta = 0 \quad ???$$

It has the center too, see the following Lectures.

## Example:

The dynamics of a point-mass pendulum is

$$\ddot{\theta} + a \cdot \sin\theta = 0$$

Its linearization around the downward equilibrium $\theta = 0$ is

$$\ddot{z} + a \cdot z = 0$$

The pendulum has an energy as the integral

$$E(\theta, \dot{\theta}) = \dot{\theta}^2 + 2a\,(1 - \cos\theta) = \dot{\theta}^2 + a \cdot \theta^2 + \ldots$$

$\Rightarrow$ Pendulum has the center around its downward position $\Leftarrow$

---

What is about the system

$$\ddot{\theta} + \left[\sin\left(100|\theta|\right) - \sqrt{|\theta|}\right]\dot{\theta}^2 + a \cdot \sin\theta = 0 \quad ???$$

It has the center too, see the following Lectures.

## Example:

The dynamics of a point-mass pendulum is

$$\ddot{\theta} + a \cdot \sin\theta = 0$$

Its linearization around the downward equilibrium $\theta = 0$ is

$$\ddot{z} + a \cdot z = 0$$

The pendulum has an energy as the integral

$$E(\theta, \dot{\theta}) = \dot{\theta}^2 + 2a\,(1 - \cos\theta) = \dot{\theta}^2 + a \cdot \theta^2 + \ldots$$

$\Rightarrow$ Pendulum has the center around its downward position $\Leftarrow$

---

What is about the system

$$\ddot{\theta} + \left[\sin(100|\theta|) - \sqrt{|\theta|}\right] \dot{\theta}^2 + a \cdot \sin\theta = 0 \quad ???$$

It has the center too, see the following Lectures.

## Example:

The dynamics of a point-mass pendulum is

$$\ddot{\theta} + a \cdot \sin\theta = 0$$

Its linearization around the downward equilibrium $\theta = 0$ is

$$\ddot{z} + a \cdot z = 0$$

The pendulum has an energy as the integral

$$E(\theta, \dot{\theta}) = \dot{\theta}^2 + 2a\,(1 - \cos\theta) = \dot{\theta}^2 + a \cdot \theta^2 + \ldots$$

$\Rightarrow$ Pendulum has the center around its downward position $\Leftarrow$

---

What is about the system

$$\ddot{\theta} + \left[\sin\left(100|\theta|\right) - \sqrt{|\theta|}\right]\dot{\theta}^2 + a \cdot \sin\theta = 0 \quad ???$$

It has the center too, see the following Lectures.

# Poincare's arguments on stability

# Poincare First Return Map

Consider the system
$$\begin{aligned}\dot{x} &= P(x,y)\\ \dot{y} &= Q(x,y)\end{aligned}$$

Suppose that there is a line $l$ transverse to its solutions.

---

Introduce a scalar parameter $s$ that uniquely defined points on $l$.

---

Suppose a trajectory $L$

- intersects $l$ in the point $s_0$ when $t = t_0$
- intersects $l$ one more time in the point $s_1$ when $t = t_1 > t_0$

---

$\Rightarrow$ There is a vicinity of $s_0$ such that solutions originated from that subset of $l$ intersect $l$ one more time, i.e. the map $f : l \to l$

$$s \mapsto \bar{s} = f(s), \quad \forall\, s \in (s_0 - \varepsilon_1,\, s_0 + \varepsilon_2)\,, \quad \varepsilon_1, \varepsilon_2 > 0$$

is well defined. It is the Poincare first return map.

# Poincare First Return Map

Consider the system
$$\begin{aligned}\dot{x} &= P(x,y)\\ \dot{y} &= Q(x,y)\end{aligned}$$

Suppose that there is a line $l$ transverse to its solutions.

---

Introduce a scalar parameter $s$ that uniquely defined points on $l$.

---

Suppose a trajectory $L$

- intersects $l$ in the point $s_0$ when $t = t_0$
- intersects $l$ one more time in the point $s_1$ when $t = t_1 > t_0$

---

$\Rightarrow$ There is a vicinity of $s_0$ such that solutions originated from that subset of $l$ intersect $l$ one more time, i.e. the map $f : l \to l$

$$s \mapsto \bar{s} = f(s), \quad \forall\, s \in (s_0 - \varepsilon_1, s_0 + \varepsilon_2)\,, \quad \varepsilon_1, \varepsilon_2 > 0$$

is well defined. It is the Poincare first return map.

# Poincare First Return Map

Consider the system
$$\begin{aligned}\dot{x} &= P(x,y)\\ \dot{y} &= Q(x,y)\end{aligned}$$

Suppose that there is a line $l$ transverse to its solutions.

---

Introduce a scalar parameter $s$ that uniquely defined points on $l$.

---

Suppose a trajectory $L$

- intersects $l$ in the point $s_0$ when $t = t_0$
- intersects $l$ one more time in the point $s_1$ when $t = t_1 > t_0$

---

$\Rightarrow$ There is a vicinity of $s_0$ such that solutions originated from that subset of $l$ intersect $l$ one more time, i.e. the map $f : l \to l$

$$s \mapsto \bar{s} = f(s), \quad \forall\, s \in (s_0 - \varepsilon_1,\, s_0 + \varepsilon_2)\,, \quad \varepsilon_1,\, \varepsilon_2 > 0$$

is well defined. It is the Poincare first return map.

# Poincare First Return Map

Consider the system

$$\begin{aligned}\dot{x} &= P(x,y)\\ \dot{y} &= Q(x,y)\end{aligned}$$

Suppose that there is a line $l$ transverse to its solutions.

---

Introduce a scalar parameter $s$ that uniquely defined points on $l$.

---

Suppose a trajectory $L$

- intersects $l$ in the point $s_0$ when $t = t_0$
- intersects $l$ one more time in the point $s_1$ when $t = t_1 > t_0$

---

$\Rightarrow$ There is a vicinity of $s_0$ such that solutions originated from that subset of $l$ intersect $l$ one more time, i.e. the map $f : l \to l$

$$s \mapsto \bar{s} = f(s), \quad \forall\, s \in (s_0 - \varepsilon_1,\, s_0 + \varepsilon_2)\,, \quad \varepsilon_1,\, \varepsilon_2 > 0$$

is well defined. It is the Poincare first return map.

## Poincare First Return Map (Cont'd)

Suppose that the trajectory $L$ intersects the line $l$ in the points

$$s_0,\ s_1,\ s_2,\ \ldots, s_k,\ \ldots$$

where $s_1 = f(s_0),\ s_2 = f(s_1),\ \ldots,\ s_k = f(s_{k-1}),\ \ldots$

Trajectory $L$ converges to a cycle $L^*$ $\Leftrightarrow$ $\lim_{k\to\infty} s_k = s^*,\ f(s^*) = s^*$

Definition: *The stationary point $s^*$ of the Poincare first return map $f(\cdot)$, $s^* = f(s^*)$, is referred to as*

- *stable, if there is its neighborhood $\mathcal{O} \subset l$, $s^* \in \mathcal{O}$, such that for any $s_0 \in \mathcal{O}$ the corresponding sequence $\{s_1, s_2, \ldots\}$ generated by $f(\cdot)$ converges to $s^*$*

$$s_k \to s^* \quad as \quad k \to +\infty$$

- *unstable, otherwise.* ■

## Poincare First Return Map (Cont'd)

Suppose that the trajectory $L$ intersects the line $l$ in the points

$$s_0,\ s_1,\ s_2,\ \ldots, s_k,\ \ldots$$

where $s_1 = f(s_0),\ s_2 = f(s_1),\ \ldots,\ s_k = f(s_{k-1}),\ \ldots$

Trajectory $L$ converges to a cycle $L^*$ $\Leftrightarrow$ $\lim_{k\to\infty} s_k = s^*,\ f(s^*) = s^*$

Definition: *The stationary point $s^*$ of the Poincare first return map $f(\cdot)$, $s^* = f(s^*)$, is referred to as*

- *stable, if there is its neighborhood $\mathcal{O} \subset l$, $s^* \in \mathcal{O}$, such that for any $s_0 \in \mathcal{O}$ the corresponding sequence $\{s_1, s_2, \ldots\}$ generated by $f(\cdot)$ converges to $s^*$*

$$s_k \to s^* \quad \text{as} \quad k \to +\infty$$

- *unstable, otherwise.* ■

# Poincare First Return Map (Cont'd)

Suppose that the trajectory $L$ intersects the line $l$ in the points

$$s_0,\ s_1,\ s_2,\ \ldots, s_k,\ \ldots$$

where $s_1 = f(s_0),\ s_2 = f(s_1),\ \ldots,\ s_k = f(s_{k-1}),\ \ldots$

---

Trajectory $L$ converges to a cycle $L^*$ $\Leftrightarrow$ $\lim_{k\to\infty} s_k = s^*,\ f(s^*) = s^*$

---

Definition: *The stationary point $s^*$ of the Poincare first return map $f(\cdot)$, $s^* = f(s^*)$, is referred to as*

- *stable, if there is its neighborhood $\mathcal{O} \subset l$, $s^* \in \mathcal{O}$, such that for any $s_0 \in \mathcal{O}$ the corresponding sequence $\{s_1, s_2, \ldots\}$ generated by $f(\cdot)$ converges to $s^*$*

$$s_k \to s^* \quad \text{as} \quad k \to +\infty$$

- *unstable, otherwise.* ■

# Stability of Poincare First Return Map

Lemma: *A stationary point $s^*$ of the Poincare first return map*

$$s_{k+1} = f(s_k)$$

*is stable, if*

$$f'(s^*) < 1,$$

*and unstable, if*

$$f'(s^*) > 1.$$

*However, if*

$$f'(s^*) = 1,$$

*then the cycle is called complex, and one needs to consider higher derivatives of the map $f(\cdot)$ at $s = s^*$.* ■

---

Check Lamerey plots of the map $\bar{s} = f(s)$ on the $(s, \bar{s})$ plane!

# Stability of Poincare First Return Map

Lemma: *A stationary point $s^*$ of the Poincare first return map*

$$s_{k+1} = f(s_k)$$

*is stable, if*

$$f'(s^*) < 1,$$

*and unstable, if*

$$f'(s^*) > 1.$$

*However, if*

$$f'(s^*) = 1,$$

*then the cycle is called complex, and one needs to consider higher derivatives of the map $f(\cdot)$ at $s = s^*$.* ■

---

Check Lamerey plots of the map $\bar{s} = f(s)$ on the $(s, \bar{s})$ plane!

# Stability of Poincare First Return Map

Lemma: *A stationary point $s^*$ of the Poincare first return map*

$$s_{k+1} = f(s_k)$$

*is stable, if*

$$f'(s^*) < 1,$$

*and unstable, if*

$$f'(s^*) > 1.$$

*However, if*

$$f'(s^*) = 1,$$

*then the cycle is called complex, and one needs to consider higher derivatives of the map $f(\cdot)$ at $s = s^*$.* ■

---

Check Lamerey plots of the map $\bar{s} = f(s)$ on the $(s, \bar{s})$ plane!

# Poincare First Return Map (Cont'd)

Given a system

$$\begin{aligned}\dot{x} &= P(x, y)\\ \dot{y} &= Q(x, y)\end{aligned}$$

Suppose that it has a $T$-periodic solution

$$x = \phi(t), \quad y = \psi(t), \quad t \in [0, T]$$

---

Q: How to find the Poincare first return map $f(\cdot)$ for this periodic solution?

---

Q: How to check the conditions of Lemma, how to compute $f'(\cdot)$?

Given a system

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned}$$

Suppose that it has a $T$-periodic solution

$$x = \phi(t), \quad y = \psi(t), \quad t \in [0, T]$$

---

Q: How to find the Poincare first return map $f(\cdot)$ for this periodic solution?

---

Q: How to check the conditions of Lemma, how to compute $f'(\cdot)$?

# Poincare First Return Map (Cont'd)

Given a system

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned}$$

Suppose that it has a $T$-periodic solution

$$x = \phi(t), \quad y = \psi(t), \quad t \in [0, T]$$

---

Q: How to find the Poincare first return map $f(\cdot)$ for this periodic solution?

---

Q: How to check the conditions of Lemma, how to compute $f'(\cdot)$?

# Poincare First Return Map (Cont'd)

Introduce new coordinates $(u, v)$ in some vicinity of the cycle $L^*$:

- $v = C$ are closed curves around $L^*$;
- $v = 0$ coincides with $L^*$;
- $u = C$ are orthogonal to $L^*$.

One can think of $(u, v)$ as similar to polar coordinates $(\theta, \rho)$.

---

If a $T$-periodic solution is given

$$x = \phi(t), \quad y = \psi(t),$$

then a transformation into the new coordinates is defined by

$$\begin{aligned} x &= \phi(u) + v\psi'(u) \\ y &= \psi(u) - v\phi'(u) \end{aligned}$$

# Poincare First Return Map (Cont'd)

Introduce new coordinates $(u, v)$ in some vicinity of the cycle $L^*$:

- $v = C$ are closed curves around $L^*$;
- $v = 0$ coincides with $L^*$;
- $u = C$ are orthogonal to $L^*$.

One can think of $(u, v)$ as similar to polar coordinates $(\theta, \rho)$.

---

If a $T$-periodic solution is given

$$x = \phi(t), \quad y = \psi(t),$$

then a transformation into the new coordinates is defined by

$$\begin{aligned} x &= \phi(u) + v\psi'(u) \\ y &= \psi(u) - v\phi'(u) \end{aligned}$$

# Poincare First Return Map (Cont'd)

If $P$, $Q$ are analytic functions, then the system

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned}$$

can be rewritten in new coordinates $(u, v)$ as one diff. equation

$$\frac{dv}{du} = \Phi(u, v) = A_1(u)v + A_2(u)v^2 + A_3(u)v^3 + \ldots$$

---

The closed trajectory $L^*$ corresponds to the solution $v = 0$.

---

Functions $A_k(u)$ can be explicitly found, for instance.

$$A_1(u) = P'_x(\phi(u), \psi(u)) + Q'_y(\phi(u), \psi(u)) - \\ - \frac{d}{du}\Big[\ln\left\{\phi'(u)^2 + \psi'(u)^2\right\}\Big]$$

## Poincare First Return Map (Cont'd)

If $P$, $Q$ are analytic functions, then the system

$$\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{aligned}$$

can be rewritten in new coordinates $(u,v)$ as one diff. equation

$$\frac{dv}{du} = \Phi(u,v) = A_1(u)v + A_2(u)v^2 + A_3(u)v^3 + \dots$$

---

The closed trajectory $L^*$ corresponds to the solution $v = 0$.

---

Functions $A_k(u)$ can be explicitly found, for instance.

$$A_1(u) = P'_x(\phi(u),\psi(u)) + Q'_y(\phi(u),\psi(u)) - \\ - \frac{d}{du}\Big[\ln\left\{\phi'(u)^2 + \psi'(u)^2\right\}\Big]$$

# Poincare First Return Map (Cont'd)

If $P$, $Q$ are analytic functions, then the system

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned}$$

can be rewritten in new coordinates $(u, v)$ as one diff. equation

$$\frac{dv}{du} = \Phi(u, v) = A_1(u)v + A_2(u)v^2 + A_3(u)v^3 + \dots$$

---

The closed trajectory $L^*$ corresponds to the solution $v = 0$.

---

Functions $A_k(u)$ can be explicitly found, for instance.

$$A_1(u) = P'_x(\phi(u), \psi(u)) + Q'_y(\phi(u), \psi(u)) - \\ - \frac{d}{du}\Big[ \ln \big\{ \phi'(u)^2 + \psi'(u)^2 \big\} \Big]$$

# Poincare First Return Map (Cont'd)

Consider a solution $v = f(u; 0, v_0)$ of the system

$$\frac{d}{du}v = \Phi(u, v) = A_1(u)v + A_2(u)v^2 + A_3(u)v^3 + \dots$$

with initial conditions in $u_0 = 0$, $v_0$. It can be written as a series

$$v = f(u; 0, v_0) = a_1(u)v_0 + a_2(u)v_0^2 + \dots + a_k(u)v_0^k + \dots$$

---

Substituting this function into the diff. equation for $v(u)$, we get

$$\frac{d}{du}\left[a_1(u)v_0 + a_2(u)v_0^2 + \dots + a_k(u)v_0^k + \dots\right] =$$
$$A_1(u)\left[a_1(u)v_0 + a_2(u)v_0^2 + \dots\right] + A_2(u)\left[a_1(u)v_0 + a_2(u)v_0^2 + \dots\right]^2 + \dots$$

or

$$a_1'(u)v_0 + a_2'(u)v_0^2 + \dots + a_k'(u)v_0^k + \dots =$$
$$A_1(u)a_1(u)v_0 + \left[A_1(u)a_2(u) + A_2(u)\left[a_1(u)\right]^2\right]v_0^2 + \dots$$

# Poincare First Return Map (Cont'd)

Consider a solution $v = f(u; 0, v_0)$ of the system

$$\frac{d}{du} v = \Phi(u, v) = A_1(u)v + A_2(u)v^2 + A_3(u)v^3 + \dots$$

with initial conditions in $u_0 = 0$, $v_0$. It can be written as a series

$$v = f(u; 0, v_0) = a_1(u)v_0 + a_2(u)v_0^2 + \dots + a_k(u)v_0^k + \dots$$

---

Substituting this function into the diff. equation for $v(u)$, we get

$$\frac{d}{du}\left[a_1(u)v_0 + a_2(u)v_0^2 + \dots + a_k(u)v_0^k + \dots\right] =$$

$$A_1(u)\left[a_1(u)v_0 + a_2(u)v_0^2 + \dots\right] + A_2(u)\left[a_1(u)v_0 + a_2(u)v_0^2 + \dots\right]^2 + \dots$$

or

$$a_1'(u)v_0 + a_2'(u)v_0^2 + \dots + a_k'(u)v_0^k + \dots =$$

$$A_1(u)a_1(u)v_0 + \left[A_1(u)a_2(u) + A_2(u)\left[a_1(u)\right]^2\right]v_0^2 + \dots$$

## Poincare First Return Map (Cont'd)

Consider a solution $v = f(u; 0, v_0)$ of the system

$$\frac{d}{du} v = \Phi(u, v) = A_1(u)v + A_2(u)v^2 + A_3(u)v^3 + \dots$$

with initial conditions in $u_0 = 0$, $v_0$. It can be written as a series

$$v = f(u; 0, v_0) = a_1(u)v_0 + a_2(u)v_0^2 + \cdots + a_k(u)v_0^k + \dots$$

---

Substituting this function into the diff. equation for $v(u)$, we get

$$\frac{d}{du}\left[a_1(u)v_0 + a_2(u)v_0^2 + \cdots + a_k(u)v_0^k + \dots\right] =$$
$$A_1(u)\left[a_1(u)v_0 + a_2(u)v_0^2 + \dots\right] + A_2(u)\left[a_1(u)v_0 + a_2(u)v_0^2 + \dots\right]^2 + \dots$$

or

$$a_1'(u)v_0 + a_2'(u)v_0^2 + \cdots + a_k'(u)v_0^k + \cdots =$$
$$A_1(u)a_1(u)v_0 + \left[A_1(u)a_2(u) + A_2(u)\left[a_1(u)\right]^2\right]v_0^2 + \dots$$

## Poincare First Return Map (Cont'd)

This gives recurrent equations on unknown functions $a_i(u)$:

$$
\begin{aligned}
a_1'(u) &= A_1(u)a_1(u), \\
a_2'(u) &= A_1(u)a_2(u) + A_2(u)\left[a_1(u)\right]^2, \\
&\vdots
\end{aligned}
$$

where the initial conditions $a_1(0)$, $a_2(0)$, ... are defined by the series

$$v_0 = f(u; 0, v_0)\Big|_{u=0} = a_1(0)v_0 + a_2(0)v_0^2 + \cdots + a_k(0)v_0^k + \ldots$$

$$\Downarrow$$

$$a_1(0) = 1, \quad a_2(0) = a_3(0) = \cdots = a_k(0) = \cdots = 0$$

## Poincare First Return Map (Cont'd)

This gives recurrent equations on unknown functions $a_i(u)$:

$$\begin{aligned} a_1'(u) &= A_1(u)a_1(u), \\ a_2'(u) &= A_1(u)a_2(u) + A_2(u)\left[a_1(u)\right]^2, \\ &\vdots \end{aligned}$$

where the initial conditions $a_1(0)$, $a_2(0)$, ... are defined by the series

$$v_0 = f(u; 0, v_0)\Big|_{u=0} = a_1(0)v_0 + a_2(0)v_0^2 + \cdots + a_k(0)v_0^k + \ldots$$

$$\Downarrow$$

$$a_1(0) = 1, \quad a_2(0) = a_3(0) = \cdots = a_k(0) = \cdots = 0$$

## Poincare First Return Map (Cont'd)

The Poincare first return map, defined on $I = \{u = 0\}$, will be

$$v = f(T;0,v_0) = a_1(T)v_0 + a_2(T)v_0^2 + \cdots + a_k(T)v_0^k + \ldots$$
$$= \alpha_1 v_0 + \alpha_2 v_0^2 + \cdots + \alpha_k v_0^k + \ldots$$

To check stability of a simple cycle we need to know only

$$\left.\frac{d\, f(T;0,v_0)}{dv_0}\right|_{v_0=0} = \alpha_1$$

If $\alpha_1 < 1$ $\Rightarrow$ the cycle is stable

If $\alpha_1 > 1$ $\Rightarrow$ the cycle is unstable

## Poincare First Return Map (Cont'd)

The Poincare first return map, defined on $I = \{u = 0\}$, will be

$$v = f(T; 0, v_0) = a_1(T)v_0 + a_2(T)v_0^2 + \cdots + a_k(T)v_0^k + \ldots$$
$$= \alpha_1 v_0 + \alpha_2 v_0^2 + \cdots + \alpha_k v_0^k + \ldots$$

---

To check stability of a simple cycle we need to know only

$$\left.\frac{d\, f(T; 0, v_0)}{dv_0}\right|_{v_0=0} = \alpha_1$$

---

If $\alpha_1 < 1 \quad \Rightarrow \quad$ the cycle is stable

If $\alpha_1 > 1 \quad \Rightarrow \quad$ the cycle is unstable

## Poincare First Return Map (Cont'd)

The Poincare first return map, defined on $I = \{u = 0\}$, will be

$$\begin{aligned} v = f(T; 0, v_0) &= a_1(T)v_0 + a_2(T)v_0^2 + \cdots + a_k(T)v_0^k + \dots \\ &= \alpha_1 v_0 + \alpha_2 v_0^2 + \cdots + \alpha_k v_0^k + \dots \end{aligned}$$

---

To check stability of a simple cycle we need to know only

$$\left. \frac{d\, f(T; 0, v_0)}{dv_0} \right|_{v_0=0} = \alpha_1$$

---

If $\alpha_1 < 1$ $\Rightarrow$ the cycle is stable

If $\alpha_1 > 1$ $\Rightarrow$ the cycle is unstable

# Poincare First Return Map (Cont'd)

The constant $\alpha_1 = a_1(T)$ is known as a multiplier of the cycle. Its value can be explicitly found:

- Integrate the equation: $a_1' = A_1(u)a_1, \quad a_1(0) = 1.$

  The answer is
  $$a_1(T) = \exp\left[\int_0^T A_1(u)du\right] a_1(0)$$

- We know that
  $$A_1(u) = P_x'(\phi(u), \psi(u)) + Q_y'(\phi(u), \psi(u)) - \\ - \frac{d}{du}\left[\ln\left\{\phi'(u)^2 + \psi'(u)^2\right\}\right]$$

  then
  $$\alpha_1 = a_1(T) = \exp\left[\int_0^T \left\{P_x'(\phi(u), \psi(u)) + Q_y'(\phi(u), \psi(u))\right\} du\right]$$

The constant $\alpha_1 = a_1(T)$ is known as a multiplier of the cycle. Its value can be explicitly found:

- Integrate the equation: $a_1' = A_1(u)a_1, \quad a_1(0) = 1.$

  The answer is
  $$a_1(T) = \exp\left[\int_0^T A_1(u)du\right] a_1(0)$$

- We know that
  $$A_1(u) = P_x'(\phi(u),\psi(u)) + Q_y'(\phi(u),\psi(u)) - \\ -\frac{d}{du}\left[\ln\left\{\phi'(u)^2 + \psi'(u)^2\right\}\right]$$

  then
  $$\alpha_1 = a_1(T) = \exp\left[\int_0^T \left\{P_x'(\phi(u),\psi(u)) + Q_y'(\phi(u),\psi(u))\right\} du\right]$$

## Poincare First Return Map (Cont'd)

The constant $\alpha_1 = a_1(T)$ is known as a multiplier of the cycle. Its value can be explicitly found:

- Integrate the equation: $a_1' = A_1(u)a_1, \quad a_1(0) = 1.$
  The answer is
  $$a_1(T) = \exp\left[\int_0^T A_1(u)du\right] a_1(0)$$
- We know that
  $$A_1(u) = P_x'(\phi(u),\psi(u)) + Q_y'(\phi(u),\psi(u)) - \\ -\frac{d}{du}\left[\ln\left\{\phi'(u)^2 + \psi'(u)^2\right\}\right]$$
  then
  $$\alpha_1 = a_1(T) = \exp\left[\int_0^T \left\{P_x'(\phi(u),\psi(u)) + Q_y'(\phi(u),\psi(u))\right\} du\right]$$

# Poincare First Return Map (Cont'd)

The constant $\alpha_1 = a_1(T)$ is known as a multiplier of the cycle. Its value can be explicitly found:

- Integrate the equation: $a_1' = A_1(u)a_1, \quad a_1(0) = 1.$

  The answer is
  $$a_1(T) = \exp\left[\int_0^T A_1(u)du\right] a_1(0)$$

- We know that
  $$A_1(u) = P_x'(\phi(u), \psi(u)) + Q_y'(\phi(u), \psi(u)) - \\ - \frac{d}{du}\left[\ln\left\{\phi'(u)^2 + \psi'(u)^2\right\}\right]$$

  then
  $$\alpha_1 = a_1(T) = \exp\left[\int_0^T \left\{P_x'(\phi(u), \psi(u)) + Q_y'(\phi(u), \psi(u))\right\} du\right]$$

## Example:

$$\begin{aligned} \dot{x} &= -y - \varepsilon x \left(x^2 + y^2 - 1\right) &= P(x, y) \\ \dot{y} &= x - \delta y \left(x^2 + y^2 - 1\right) &= Q(x, y) \end{aligned}$$

The system has a cycle

$$x = \phi(u) = \cos u, \quad y = \psi(u) = \sin u, \quad 0 \le u \le 2\pi$$

The verify (in)stability of this cycle, one can compute

$$\alpha_1 = \exp\left[\int_0^{2\pi} \left\{ P'_x(\phi(u), \psi(u)) + Q'_y(\phi(u), \psi(u)) \right\} du\right]$$

$$= \exp\left[\int_0^{2\pi} \left\{ -2\varepsilon \cos^2(u) - 2\delta \sin^2(u) \right\} du\right] = \exp\left[-2\pi(\varepsilon + \delta)\right]$$

---

If $(\varepsilon + \delta) > 0 \quad \Rightarrow \quad$ the cycle is orbitally stable

If $(\varepsilon + \delta) < 0 \quad \Rightarrow \quad$ the cycle is unstable

# Example:

$$\begin{aligned}\dot{x} &= -y - \varepsilon x\left(x^2 + y^2 - 1\right) &= P(x, y)\\ \dot{y} &= x - \delta y\left(x^2 + y^2 - 1\right) &= Q(x, y)\end{aligned}$$

The system has a cycle

$$x = \phi(u) = \cos u, \quad y = \psi(u) = \sin u, \quad 0 \le u \le 2\pi$$

The verify (in)stability of this cycle, one can compute

$$\alpha_1 = \exp\left[\int_0^{2\pi} \left\{P'_x(\phi(u), \psi(u)) + Q'_y(\phi(u), \psi(u))\right\} du\right]$$

$$= \exp\left[\int_0^{2\pi} \left\{-2\varepsilon\cos^2(u) - 2\delta\sin^2(u)\right\} du\right] = \exp\left[-2\pi(\varepsilon + \delta)\right]$$

---

If $(\varepsilon + \delta) > 0$ $\Rightarrow$ the cycle is orbitally stable

If $(\varepsilon + \delta) < 0$ $\Rightarrow$ the cycle is unstable

## Example:

$$\begin{aligned}\dot{x} &= -y - \varepsilon x\left(x^2 + y^2 - 1\right) &= P(x,y)\\ \dot{y} &= x - \delta y\left(x^2 + y^2 - 1\right) &= Q(x,y)\end{aligned}$$

The system has a cycle

$$x = \phi(u) = \cos u, \quad y = \psi(u) = \sin u, \quad 0 \le u \le 2\pi$$

The verify (in)stability of this cycle, one can compute

$$\alpha_1 = \exp\left[\int_0^{2\pi}\left\{P'_x(\phi(u),\psi(u)) + Q'_y(\phi(u),\psi(u))\right\} du\right]$$

$$= \exp\left[\int_0^{2\pi}\left\{-2\varepsilon\cos^2(u) - 2\delta\sin^2(u)\right\} du\right] = \exp\left[-2\pi(\varepsilon+\delta)\right]$$

If $(\varepsilon+\delta) > 0 \quad \Rightarrow \quad$ the cycle is orbitally stable

If $(\varepsilon+\delta) < 0 \quad \Rightarrow \quad$ the cycle is unstable

## Example:

$$\begin{aligned}\dot{x} &= -y - \varepsilon x\left(x^2 + y^2 - 1\right) &= P(x,y)\\ \dot{y} &= x - \delta y\left(x^2 + y^2 - 1\right) &= Q(x,y)\end{aligned}$$

The system has a cycle

$$x = \phi(u) = \cos u, \quad y = \psi(u) = \sin u, \quad 0 \leq u \leq 2\pi$$

The verify (in)stability of this cycle, one can compute

$$\alpha_1 = \exp\left[\int_0^{2\pi} \left\{P'_x(\phi(u), \psi(u)) + Q'_y(\phi(u), \psi(u))\right\} du\right]$$

$$= \exp\left[\int_0^{2\pi}\left\{-2\varepsilon\cos^2(u) - 2\delta\sin^2(u)\right\} du\right] = \exp\left[-2\pi(\varepsilon + \delta)\right]$$

---

If $(\varepsilon + \delta) > 0 \quad \Rightarrow \quad$ the cycle is orbitally stable

If $(\varepsilon + \delta) < 0 \quad \Rightarrow \quad$ the cycle is unstable

# Small Parameter Methods

## Small Parameter Methods

One way for finding periodic solutions is based on:

Closeness (in some sense) of our system to another one, which possesses a cycle

---

Consider a system

$$\dot{x} = X_0(x) + \mu X(\mu, x) = X_0(x) + \mu X_1(x) + \mu^2 X_2(x) + \dots$$
$$= \sum_{k=0}^{+\infty} \mu^k X_k(x)$$

where $\mu$ is small parameter; $x$ is state vector. It is assumed that the system

$$\dot{x} = X_0(x)$$

has $T$-periodic solution $x_{(0)} = x_{(0)}(t)$.

---

When does the system with $\mu \neq 0$ have a cycle?

## Small Parameter Methods

One way for finding periodic solutions is based on:

Closeness (in some sense) of our system to another one, which possesses a cycle

---

Consider a system

$$\begin{aligned} \dot{x} &= X_0(x) + \mu X(\mu, x) = X_0(x) + \mu X_1(x) + \mu^2 X_2(x) + \dots \\ &= \sum_{k=0}^{+\infty} \mu^k X_k(x) \end{aligned}$$

where $\mu$ is small parameter; $x$ is state vector. It is assumed that the system

$$\dot{x} = X_0(x)$$

has $T$-periodic solution $x_{(0)} = x_{(0)}(t)$.

---

When does the system with $\mu \neq 0$ have a cycle?

One way for finding periodic solutions is based on:

Closeness (in some sense) of our system to another one, which possesses a cycle

---

Consider a system

$$\begin{aligned}\dot{x} &= X_0(x) + \mu X(\mu, x) = X_0(x) + \mu X_1(x) + \mu^2 X_2(x) + \dots \\ &= \sum_{k=0}^{+\infty} \mu^k X_k(x)\end{aligned}$$

where $\mu$ is small parameter; $x$ is state vector. It is assumed that the system

$$\dot{x} = X_0(x)$$

has $T$-periodic solution $x_{(0)} = x_{(0)}(t)$.

---

When does the system with $\mu \neq 0$ have a cycle?

# Small Parameter Methods (Cont'd)

It is suggested to search a periodic solution $x_*(t)$ in the form of series

$$x_*(t) = x_{(0)}(t) + \mu x_{(1)}(t) + \mu^2 x_{(2)}(t) + \cdots = \sum_{k=0}^{+\infty} \mu^k x_{(k)}(t),$$

where functions $x_{(k)}(\cdot)$ are chosen somehow.

Precautions:

- Even the nominal solution $x_{(0)}(t)$ is of period $T$, the function $x_*(t)$ can be of different (unknown) period

$$x_*(t) = x_*\left(t + T + \triangle(\mu)\right);$$

- functions $x_{(k)}(t)$ are not necessarily periodic, but if periodic, they can be of different periods.

## Small Parameter Methods (Cont'd)

It is suggested to search a periodic solution $x_*(t)$ in the form of series

$$x_*(t) = x_{(0)}(t) + \mu x_{(1)}(t) + \mu^2 x_{(2)}(t) + \cdots = \sum_{k=0}^{+\infty} \mu^k x_{(k)}(t),$$

where functions $x_{(k)}(\cdot)$ are chosen somehow.

---

Precautions:

- Even the nominal solution $x_{(0)}(t)$ is of period $T$, the function $x_*(t)$ can be of different (unknown) period
$$x_*(t) = x_*\Big(t + T + \triangle(\mu)\Big);$$
- functions $x_{(k)}(t)$ are not necessarily periodic, but if periodic, they can be of different periods.

# Small Parameter Methods (Cont'd)

It is suggested to search a periodic solution $x_*(t)$ in the form of series

$$x_*(t) = x_{(0)}(t) + \mu x_{(1)}(t) + \mu^2 x_{(2)}(t) + \cdots = \sum_{k=0}^{+\infty} \mu^k x_{(k)}(t),$$

where functions $x_{(k)}(\cdot)$ are chosen somehow.

---

Precautions:

- Even the nominal solution $x_{(0)}(t)$ is of period $T$, the function $x_*(t)$ can be of different (unknown) period

$$x_*(t) = x_*\left(t + T + \triangle(\mu)\right);$$

- functions $x_{(k)}(t)$ are not necessarily periodic, but if periodic, they can be of different periods.

## Example:

Consider an expansion of $\sin(\omega+\mu)t$ in powers of $\mu$:

$$\sin(\omega+\mu)t = \Big[\sin(\omega+\mu)t\Big]\Big|_{\mu=0} + \mu\,\Big[\sin(\omega+\mu)t\Big]'_{\mu}\Big|_{\mu=0} + \mu^2 \dots$$

$$= \underbrace{\sin\omega t}_{=x_{(0)}(t)} + \underbrace{\mu t\cos\omega t}_{\text{it grows!}} - \frac{1}{2}\underbrace{(\mu t)^2\sin\omega t}_{\text{it grows!}} + \dots$$

The terms in the expansion with factors $\mu t$, $(\mu t)^2$, ... are known as secular terms in celestial mechanics. They grow without a bound with time for any $\mu \neq 0$!

---

Taking into account only a few terms in series always BAD!

Such approximation can be only used for finite time interval $[0, \tau]$ with, let say, $\tau \approx \frac{1}{\mu}$

## Example:

Consider an expansion of $\sin(\omega+\mu)t$ in powers of $\mu$:

$$\sin(\omega+\mu)t = \Big[\sin(\omega+\mu)t\Big]\Big|_{\mu=0} + \mu\,\Big[\sin(\omega+\mu)t\Big]'_{\mu}\Big|_{\mu=0} + \mu^2\ldots$$

$$= \underbrace{\sin\omega t}_{=x_{(0)}(t)} + \underbrace{\mu t\cos\omega t}_{\text{it grows!}} - \frac{1}{2}\underbrace{(\mu t)^2\sin\omega t}_{\text{it grows!}} + \ldots$$

The terms in the expansion with factors $\mu t$, $(\mu t)^2$, ... are known as secular terms in celestial mechanics. They grow without a bound with time for any $\mu \neq 0$!

Taking into account only a few terms in series always BAD!

Such approximation can be only used for finite time interval $[0, \tau]$ with, let say, $\tau \approx \frac{1}{\mu}$

# Example:

Consider an expansion of $\sin(\omega + \mu)t$ in powers of $\mu$:

$$\sin(\omega + \mu)t = \Big[\sin(\omega + \mu)t\Big]\Big|_{\mu=0} + \mu\,\Big[\sin(\omega + \mu)t\Big]'_{\mu}\Big|_{\mu=0} + \mu^2 \ldots$$

$$= \underbrace{\sin \omega t}_{= x_{(0)}(t)} + \underbrace{\mu t \cos \omega t}_{\text{it grows!}} - \frac{1}{2}\underbrace{(\mu t)^2 \sin \omega t}_{\text{it grows!}} + \ldots$$

The terms in the expansion with factors $\mu t$, $(\mu t)^2$, ... are known as secular terms in celestial mechanics. They grow without a bound with time for any $\mu \neq 0$!

---

Taking into account only a few terms in series always BAD!

Such approximation can be only used for finite time interval $[0, \tau]$ with, let say, $\tau \approx \frac{1}{\mu}$

## Example (Rayleigh system):

$$\ddot{x} + \omega^2 x = \mu \left(1 - \dot{x}^2\right) \dot{x}$$

Let us find a periodic solution of the system as a formal series

$$x_*(t) = x_{(0)}(t) + \mu x_{(1)}(t) + \mu^2 x_{(2)}(t) + \dots$$

To find functions $x_{(k)}$, one plugs-in this series into the equation

$$\frac{d^2}{dt^2}\left[x_{(0)}(t)+\mu x_{(1)}(t)+\dots\right]+\omega^2\left[x_{(0)}(t)+\mu x_{(1)}(t)+\dots\right]=$$

$$\mu\left(1-\left\{\frac{d}{dt}\left[x_{(0)}(t)+\mu x_{(1)}(t)+\dots\right]\right\}^2\right)\frac{d}{dt}\left[x_{(0)}(t)+\mu x_{(1)}(t)+\dots\right]$$

Collecting terms in powers of $\mu$, one gets

$$\mu^0\left(\ddot{x}_{(0)}+\omega^2 x_{(0)}\right)+\mu^1\left(\ddot{x}_{(1)}+\omega^2 x_{(1)}-\dot{x}_{(0)}+\dot{x}_{(0)}^3\right)+$$

$$+\mu^2\left(\ddot{x}_{(2)}+\omega^2 x_{(2)}-\dot{x}_{(1)}+3\dot{x}_{(0)}\dot{x}_{(1)}\right)+\dots=0$$

# Example (Rayleigh system):

$$\ddot{x} + \omega^2 x = \mu \left(1 - \dot{x}^2\right) \dot{x}$$

Let us find a periodic solution of the system as a formal series

$$x_*(t) = x_{(0)}(t) + \mu x_{(1)}(t) + \mu^2 x_{(2)}(t) + \ldots$$

---

To find functions $x_{(k)}$, one plugs-in this series into the equation

$$\frac{d^2}{dt^2}\left[x_{(0)}(t) + \mu x_{(1)}(t) + \ldots\right] + \omega^2\left[x_{(0)}(t) + \mu x_{(1)}(t) + \ldots\right] =$$
$$\mu\left(1 - \left\{\frac{d}{dt}\left[x_{(0)}(t) + \mu x_{(1)}(t) + \ldots\right]\right\}^2\right)\frac{d}{dt}\left[x_{(0)}(t) + \mu x_{(1)}(t) + \ldots\right]$$

---

Collecting terms in powers of $\mu$, one gets

$$\mu^0\left(\ddot{x}_{(0)} + \omega^2 x_{(0)}\right) + \mu^1\left(\ddot{x}_{(1)} + \omega^2 x_{(1)} - \dot{x}_{(0)} + \dot{x}_{(0)}{}^3\right) +$$
$$+ \mu^2\left(\ddot{x}_{(2)} + \omega^2 x_{(2)} - \dot{x}_{(1)} + 3\dot{x}_{(0)}\dot{x}_{(1)}\right) + \cdots = 0$$

## Example (Rayleigh system):

$$\ddot{x} + \omega^2 x = \mu \left(1 - \dot{x}^2\right) \dot{x}$$

Let us find a periodic solution of the system as a formal series

$$x_*(t) = x_{(0)}(t) + \mu x_{(1)}(t) + \mu^2 x_{(2)}(t) + \dots$$

---

To find functions $x_{(k)}$, one plugs-in this series into the equation

$$\frac{d^2}{dt^2}\left[x_{(0)}(t) + \mu x_{(1)}(t) + \dots\right] + \omega^2\left[x_{(0)}(t) + \mu x_{(1)}(t) + \dots\right] =$$

$$\mu\left(1 - \left\{\frac{d}{dt}\left[x_{(0)}(t) + \mu x_{(1)}(t) + \dots\right]\right\}^2\right)\frac{d}{dt}\left[x_{(0)}(t) + \mu x_{(1)}(t) + \dots\right]$$

---

Collecting terms in powers of $\mu$, one gets

$$\mu^0\left(\ddot{x}_{(0)} + \omega^2 x_{(0)}\right) + \mu^1\left(\ddot{x}_{(1)} + \omega^2 x_{(1)} - \dot{x}_{(0)} + \dot{x}_{(0)}{}^3\right) +$$

$$+\mu^2\left(\ddot{x}_{(2)} + \omega^2 x_{(2)} - \dot{x}_{(1)} + 3\dot{x}_{(0)}\dot{x}_{(1)}\right) + \dots = 0$$

## Example (Cont'd):

The recurrent differential equations for functions $x_{(k)}$ are:

$$
\begin{array}{lll}
\mu^0: & \ddot{x}_{(0)} + \omega^2 x_{(0)} = 0, & x_{(0)}(0) = A, \quad \dot{x}_{(0)}(0) = 0 \\
\mu^1: & \ddot{x}_{(1)} + \omega^2 x_{(1)} - \dot{x}_{(0)} + \dot{x}_{(0)}^3 = 0, & x_{(1)}(0) = 0, \quad \dot{x}_{(1)}(0) = 0 \\
\mu^2: & \ddot{x}_{(2)} + \omega^2 x_{(2)} + \cdots = 0, & x_{(2)}(0) = 0, \quad \dot{x}_{(2)}(0) = 0 \\
 & \vdots &
\end{array}
$$

Then

$$
\begin{array}{lrl}
\mu^0: & x_{(0)}(t) & = A\cos\omega t \\
\mu^1: & \ddot{x}_{(1)} + \omega^2 x_{(1)} & = \dot{x}_{(0)} - \dot{x}_{(0)}^3 \\
 & & = A\omega\left(\frac{3}{4}A^2\omega^2 - 1\right)\sin\omega t - \frac{1}{4}A^3\omega^3\sin 3\omega t
\end{array}
$$

The function $x_{(1)}$ can grow linearly ($\approx ct$) due to resonance! It is the secular term in the expansion for $x_*$.

## Example (Cont'd):

The recurrent differential equations for functions $x_{(k)}$ are:

$$\begin{array}{lll}
\mu^0: & \ddot{x}_{(0)} + \omega^2 x_{(0)} = 0, & x_{(0)}(0) = A, \quad \dot{x}_{(0)}(0) = 0 \\
\mu^1: & \ddot{x}_{(1)} + \omega^2 x_{(1)} - \dot{x}_{(0)} + \dot{x}_{(0)}^3 = 0, & x_{(1)}(0) = 0, \quad \dot{x}_{(1)}(0) = 0 \\
\mu^2: & \ddot{x}_{(2)} + \omega^2 x_{(2)} + \cdots = 0, & x_{(2)}(0) = 0, \quad \dot{x}_{(2)}(0) = 0 \\
 & \vdots &
\end{array}$$

---

Then

$$\begin{array}{lrl}
\mu^0: & x_{(0)}(t) & = A \cos \omega t \\
\mu^1: & \ddot{x}_{(1)} + \omega^2 x_{(1)} & = \dot{x}_{(0)} - \dot{x}_{(0)}^3 \\
 & & = A\omega \left(\frac{3}{4} A^2 \omega^2 - 1\right) \sin \omega t - \frac{1}{4} A^3 \omega^3 \sin 3\omega t
\end{array}$$

The function $x_{(1)}$ can grow linearly ($\approx ct$) due to resonance! It is the secular term in the expansion for $x_*$.

## Example (Cont'd):

To avoid secular terms, let us allow the frequency $\omega$ of cycle of perturbed system to change in recurrent computations:

$$p^2 = \omega^2 + \mu h_1 + \mu^2 h_2 + \dots$$

To find functions $x_{(k)}$, one plugs-in this series into the equation

$$\frac{d^2}{dt^2}\left[x_{(0)}+\mu x_{(1)}+\dots\right]+\omega^2\left[x_{(0)}+\mu x_{(1)}+\dots\right]=$$

$$\frac{d^2}{dt^2}\left[x_{(0)}+\mu x_{(1)}+\dots\right]+\left[p^2-\mu h_1-\mu^2 h_2-\dots\right]\left[x_{(0)}+\mu x_{(1)}+\dots\right]=$$

$$\mu\left(1-\left\{\frac{d}{dt}\left[x_{(0)}+\mu x_{(1)}+\dots\right]\right\}^2\right)\frac{d}{dt}\left[x_{(0)}+\mu x_{(1)}+\dots\right]$$

Parameters

$$h_1, \quad h_2, \quad \dots$$

will be chosen to cancel secular terms in the expansion!

## Example (Cont'd):

To avoid secular terms, let us allow the frequency $\omega$ of cycle of perturbed system to change in recurrent computations:

$$p^2 = \omega^2 + \mu h_1 + \mu^2 h_2 + \dots$$

---

To find functions $x_{(k)}$, one plugs-in this series into the equation

$$\frac{d^2}{dt^2}\left[x_{(0)} + \mu x_{(1)} + \dots\right] + \omega^2\left[x_{(0)} + \mu x_{(1)} + \dots\right] =$$

$$\frac{d^2}{dt^2}\left[x_{(0)} + \mu x_{(1)} + \dots\right] + \left[p^2 - \mu h_1 - \mu^2 h_2 - \dots\right]\left[x_{(0)} + \mu x_{(1)} + \dots\right] =$$

$$\mu\left(1 - \left\{\frac{d}{dt}\left[x_{(0)} + \mu x_{(1)} + \dots\right]\right\}^2\right)\frac{d}{dt}\left[x_{(0)} + \mu x_{(1)} + \dots\right]$$

Parameters

$$h_1, \quad h_2, \quad \dots$$

will be chosen to cancel secular terms in the expansion!

## Example (Cont'd):

To avoid secular terms, let us allow the frequency $\omega$ of cycle of perturbed system to change in recurrent computations:

$$p^2 = \omega^2 + \mu h_1 + \mu^2 h_2 + \dots$$

---

To find functions $x_{(k)}$, one plugs-in this series into the equation

$$\frac{d^2}{dt^2}\Big[x_{(0)} + \mu x_{(1)} + \dots\Big] + \omega^2 \Big[x_{(0)} + \mu x_{(1)} + \dots\Big] =$$

$$\frac{d^2}{dt^2}\Big[x_{(0)} + \mu x_{(1)} + \dots\Big] + \Big[p^2 - \mu h_1 - \mu^2 h_2 - \dots\Big]\Big[x_{(0)} + \mu x_{(1)} + \dots\Big] =$$

$$\mu\left(1 - \left\{\frac{d}{dt}\Big[x_{(0)} + \mu x_{(1)} + \dots\Big]\right\}^2\right)\frac{d}{dt}\Big[x_{(0)} + \mu x_{(1)} + \dots\Big]$$

---

Parameters

$$h_1, \quad h_2, \quad \dots$$

will be chosen to cancel secular terms in the expansion!

# Example (Cont'd):

The recurrent differential equations for functions $x_{(k)}$ are

$$
\begin{array}{lll}
\mu^0: & \ddot{x}_{(0)} + p^2 x_{(0)} = 0, & p^2 = \omega^2 \\
\mu^1: & \ddot{x}_{(1)} + p^2 x_{(1)} - h_1 x_{(0)} - \dot{x}_{(0)} + \dot{x}_{(0)}^3 = 0, & p^2 = \omega^2 + \mu h_1 \\
\mu^2: & \ddot{x}_{(2)} + p^2 x_{(2)} + \cdots = 0, & p^2 = \omega^2 + \mu h_1 + \mu^2 h_2 \\
 & \ldots & \ldots
\end{array}
$$

If the initial conditions are $x_{(0)}(0) = A$, $\dot{x}_{(0)}(0) = 0$, then

$$x_{(0)}(t) = A \cos pt$$

$x_{(1)}$ is a solution of next eqn with $x_{(1)}(0) = 0$, $\dot{x}_{(1)}(0) = 0$

$$
\begin{aligned}
\ddot{x}_{(1)} + p^2 x_{(1)} &= h_1 x_{(0)} + \dot{x}_{(0)} - \dot{x}_{(0)}^3 \\
&= \underbrace{h_1 A \cos pt}_{=0} + \underbrace{Ap\left(\tfrac{3}{4}A^2 p^2 - 1\right)\sin pt}_{=0} - \tfrac{1}{4}A^3 p^3 \sin 3pt
\end{aligned}
$$

Then $h_1 = 0$, $A = 2/p\sqrt{3}$, and $p^2 = \omega^2 + \mu h_1 \Rightarrow A = 2/\omega\sqrt{3}$

## Example (Cont'd):

The recurrent differential equations for functions $x_{(k)}$ are

$$
\begin{array}{lll}
\mu^0: & \ddot{x}_{(0)} + p^2 x_{(0)} = 0, & p^2 = \omega^2 \\
\mu^1: & \ddot{x}_{(1)} + p^2 x_{(1)} - h_1 x_{(0)} - \dot{x}_{(0)} + \dot{x}_{(0)}^3 = 0, & p^2 = \omega^2 + \mu h_1 \\
\mu^2: & \ddot{x}_{(2)} + p^2 x_{(2)} + \cdots = 0, & p^2 = \omega^2 + \mu h_1 + \mu^2 h_2 \\
 & \ldots & \ldots
\end{array}
$$

If the initial conditions are $x_{(0)}(0) = A$, $\dot{x}_{(0)}(0) = 0$, then

$$x_{(0)}(t) = A\cos pt$$

$x_{(1)}$ is a solution of next eqn with $x_{(1)}(0) = 0$, $\dot{x}_{(1)}(0) = 0$

$$
\begin{aligned}
\ddot{x}_{(1)} + p^2 x_{(1)} &= h_1 x_{(0)} + \dot{x}_{(0)} - \dot{x}_{(0)}^3 \\
&= \underbrace{h_1 A\cos pt}_{=0} + \underbrace{Ap\left(\tfrac{3}{4}A^2p^2 - 1\right)\sin pt}_{=0} - \tfrac{1}{4}A^3p^3\sin 3pt
\end{aligned}
$$

Then $h_1 = 0$, $A = 2/p\sqrt{3}$, and $p^2 = \omega^2 + \mu h_1 \Rightarrow A = 2/\omega\sqrt{3}$

## Example (Cont'd):

The recurrent differential equations for functions $x_{(k)}$ are

$$
\begin{array}{lll}
\mu^0: & \ddot{x}_{(0)} + p^2 x_{(0)} = 0, & p^2 = \omega^2 \\
\mu^1: & \ddot{x}_{(1)} + p^2 x_{(1)} - h_1 x_{(0)} - \dot{x}_{(0)} + \dot{x}_{(0)}^3 = 0, & p^2 = \omega^2 + \mu h_1 \\
\mu^2: & \ddot{x}_{(2)} + p^2 x_{(2)} + \cdots = 0, & p^2 = \omega^2 + \mu h_1 + \mu^2 h_2 \\
 & \ldots & \ldots
\end{array}
$$

If the initial conditions are $x_{(0)}(0) = A$, $\dot{x}_{(0)}(0) = 0$, then

$$x_{(0)}(t) = A \cos pt$$

$x_{(1)}$ is a solution of next eqn with $x_{(1)}(0) = 0$, $\dot{x}_{(1)}(0) = 0$

$$
\begin{aligned}
\ddot{x}_{(1)} + p^2 x_{(1)} &= h_1 x_{(0)} + \dot{x}_{(0)} - \dot{x}_{(0)}^3 \\
&= \underbrace{h_1 A \cos pt}_{=0} + \underbrace{Ap\left(\tfrac{3}{4} A^2 p^2 - 1\right) \sin pt}_{=0} - \tfrac{1}{4} A^3 p^3 \sin 3pt
\end{aligned}
$$

Then $h_1 = 0$, $A = 2/p\sqrt{3}$, and $p^2 = \omega^2 + \mu h_1 \Rightarrow A = 2/\omega\sqrt{3}$

# Example (Cont'd):

The recurrent differential equations for functions $x_{(k)}$ are

$$\begin{aligned}
\mu^0: &\quad \ddot{x}_{(0)} + p^2 x_{(0)} = 0, && p^2 = \omega^2 \\
\mu^1: &\quad \ddot{x}_{(1)} + p^2 x_{(1)} - h_1 x_{(0)} - \dot{x}_{(0)} + \dot{x}_{(0)}^3 = 0, && p^2 = \omega^2 + \mu h_1 \\
\mu^2: &\quad \ddot{x}_{(2)} + p^2 x_{(2)} + \cdots = 0, && p^2 = \omega^2 + \mu h_1 + \mu^2 h_2 \\
&\quad \ldots && \ldots
\end{aligned}$$

If the initial conditions are $x_{(0)}(0) = A$, $\dot{x}_{(0)}(0) = 0$, then

$$x_{(0)}(t) = A \cos pt$$

$x_{(1)}$ is a solution of next eqn with $x_{(1)}(0) = 0$, $\dot{x}_{(1)}(0) = 0$

$$\begin{aligned}
\ddot{x}_{(1)} + p^2 x_{(1)} &= h_1 x_{(0)} + \dot{x}_{(0)} - \dot{x}_{(0)}^3 \\
&= \underbrace{h_1 A \cos pt}_{=0} + \underbrace{Ap\left(\tfrac{3}{4}A^2p^2 - 1\right)\sin pt}_{=0} - \tfrac{1}{4}A^3p^3 \sin 3pt
\end{aligned}$$

Then $h_1 = 0$, $A = 2/p\sqrt{3}$, and $p^2 = \omega^2 + \mu h_1 \Rightarrow A = 2/\omega\sqrt{3}$

# Example (Cont'd):

Solving the differential equation for $x_{(1)}$, one gets

$$x_{(1)}(t) = \frac{A^3 p}{32}\left[\sin 3pt - 3\sin pt\right], \quad A = 2/p\sqrt{3}, \quad p = \omega$$

So that the 1st order approximation of periodic solution $x_*(t)$ is

$$\begin{aligned} x_*(t) &\approx x_{(0)}(t) + \mu\, x_{(1)}(t) \\ &= \frac{2}{\sqrt{3}\omega}\cos\omega t + \frac{\mu}{12\sqrt{3}\omega^2}\left[\sin 3\omega t - 3\sin\omega t\right] \end{aligned}$$

# Example (Cont'd):

Solving the differential equation for $x_{(1)}$, one gets

$$x_{(1)}(t) = \frac{A^3 p}{32}\left[\sin 3pt - 3\sin pt\right], \quad A = 2/p\sqrt{3}, \quad p = \omega$$

---

So that the 1st order approximation of periodic solution $x_*(t)$ is

$$\begin{aligned} x_*(t) &\approx x_{(0)}(t) + \mu\, x_{(1)}(t) \\ &= \frac{2}{\sqrt{3}\omega}\cos\omega t + \frac{\mu}{12\sqrt{3}\omega^2}\left[\sin 3\omega t - 3\sin\omega t\right] \end{aligned}$$

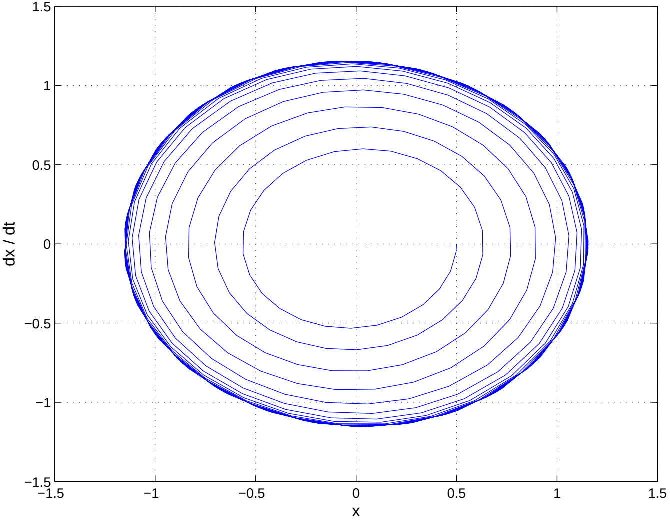


**Figure 1:** simulated periodic solution $x_*(t)$ of the Rayleigh's system when $\omega = 1$ and $\mu = 0.1$.

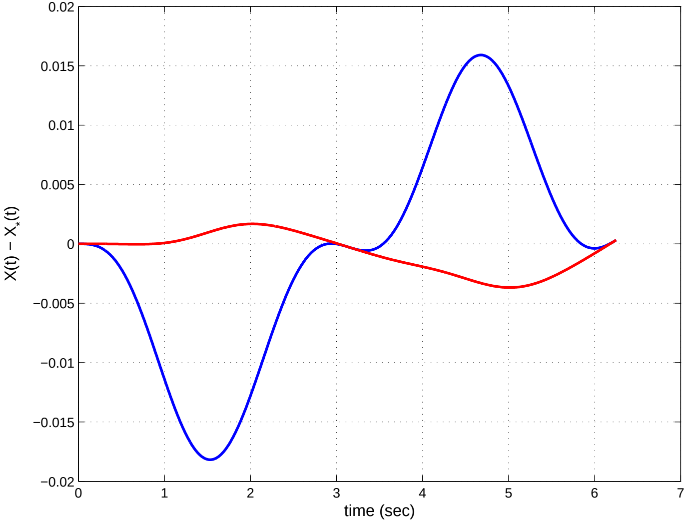


**Figure 2:** Errors between the cycle $x_*(t)$ simulated over the period and its 0-order and 1st-order approximations, when $\omega = 1$ and $\mu = 0.1$.

## Example (Cont'd):

To improve an accuracy, one can consider next functions $x_{(k)}$ in the expansion. For instance, to derive the 2nd order approximation

$$x_*(t) \approx x_{(0)}(t) + \mu\, x_{(1)}(t) + \mu^2\, x_{(2)}(t).$$

---

For that one needs to solve the next recurrent equation

$$\begin{aligned}
\ddot{x}_{(2)} + p^2 x_{(2)} &= \underbrace{h_1 x_{(1)}}_{=0} + h_2 x_{(0)} + \dot{x}_{(1)} - 3\dot{x}_{(0)}\dot{x}_{(1)} \\
&= \underbrace{A\left(h_2 - \frac{3}{32}A^2p^2 + \frac{21}{128}A^4p^4\right)\cos pt}_{=0} + \\
&\quad + A^3p^2\left(\tfrac{3}{32} - \tfrac{21}{128}A^2p^2\right)\cos 3pt + \tfrac{9}{128}A^5p^4\cos 5pt \\
p^2 &= \omega^2 + \mu h_1 + \mu^2 h_2 = \omega^2 - \frac{\mu^2}{6}
\end{aligned}$$

## Example (Cont'd):

To improve an accuracy, one can consider next functions $x_{(k)}$ in the expansion. For instance, to derive the 2nd order approximation

$$x_*(t) \approx x_{(0)}(t) + \mu\, x_{(1)}(t) + \mu^2\, x_{(2)}(t).$$

---

For that one needs to solve the next recurrent equation

$$\begin{aligned}
\ddot{x}_{(2)} + p^2 x_{(2)} &= \underbrace{h_1 x_{(1)}}_{=0} + h_2 x_{(0)} + \dot{x}_{(1)} - 3\dot{x}_{(0)}\dot{x}_{(1)} \\
&= \underbrace{A\left(h_2 - \frac{3}{32}A^2p^2 + \frac{21}{128}A^4p^4\right)\cos pt}_{=0} + \\
&\quad + A^3p^2\left(\tfrac{3}{32} - \tfrac{21}{128}A^2p^2\right)\cos 3pt + \tfrac{9}{128}A^5p^4\cos 5pt \\
p^2 &= \omega^2 + \mu h_1 + \mu^2 h_2 = \omega^2 - \frac{\mu^2}{6}
\end{aligned}$$

# Krylov-Bogolyubov Method

Consider the system

$$\ddot{x} + \omega^2 x = \mu f(x, \dot{x}),$$

The previous approach is to approximate periodic solution as

$$\begin{aligned} x_*(t) &\approx x_{(0)}(t) + \mu x_{(1)}(t) + \mu^2 x_{(2)}(t) + \cdots + \mu^k x_{(k)}(t) \\ &= A\cos pt + \mu x_{(1)}(t) + \mu^2 x_{(2)}(t) + \cdots + \mu^k x_{(k)}(t) \end{aligned}$$

---

Krylov and Bololyubov suggested in 1930s the next modification

$$x_*(t) \approx a\cos\psi + \mu x_{(1)}(t) + \mu^2 x_{(2)}(t) + \cdots + \mu^k x_{(k)}(t)$$

where $a$ and $\psi$ are functions defined by equations

$$\begin{aligned} \dot{a} &= \mu A_1(a) + \mu^2 A_2(a) + \cdots + \mu^n A_n(a) \\ \dot{\psi} &= \omega + \mu B_1(a) + \mu^2 B_2(a) + \cdots + \mu^n B_n(a) \end{aligned}$$

# Krylov-Bogolyubov Method

Consider the system

$$\ddot{x} + \omega^2 x = \mu f(x, \dot{x}),$$

The previous approach is to approximate periodic solution as

$$\begin{aligned} x_*(t) &\approx x_{(0)}(t) + \mu x_{(1)}(t) + \mu^2 x_{(2)}(t) + \cdots + \mu^k x_{(k)}(t) \\ &= A \cos pt + \mu x_{(1)}(t) + \mu^2 x_{(2)}(t) + \cdots + \mu^k x_{(k)}(t) \end{aligned}$$

---

Krylov and Bololyubov suggested in 1930s the next modification

$$x_*(t) \approx a \cos \psi + \mu x_{(1)}(t) + \mu^2 x_{(2)}(t) + \cdots + \mu^k x_{(k)}(t)$$

where $a$ and $\psi$ are functions defined by equations

$$\begin{aligned} \dot{a} &= \mu A_1(a) + \mu^2 A_2(a) + \cdots + \mu^n A_n(a) \\ \dot{\psi} &= \omega + \mu B_1(a) + \mu^2 B_2(a) + \cdots + \mu^n B_n(a) \end{aligned}$$

# Example (Van der Pol equation)

$$\ddot{x} + \omega^2 x = \mu f(x, \dot{x}) = \mu(1 - x^2)\dot{x}$$

As known, the system has an exponentially orbitally stable periodic solution $x_*(t)$ for $\mu > 0$. Let us try to approximate it as

$$x_*(t) = a(t) \cos \psi(t), \quad \dot{a}(t) = \mu A_1(a(t)), \; \dot{\psi}(t) = \omega + \mu B_1(a(t))$$

where the functions $A_1(\cdot)$, $B_1(\cdot)$ are to be found.

---

Step 1: To get the first equation relating $A_1(\cdot)$, $B_1(\cdot)$, one can postulate the form of $\dot{x}_*$ as

$$\begin{aligned} \dot{x}_* &= -a\omega \sin \psi = \dot{a} \cos \psi - a \sin \psi \, \dot{\psi} \\ &= \mu A_1(a) \cos \psi - a\omega \sin \psi - \mu a B_1(a) \sin \psi \end{aligned}$$

$$\Downarrow$$

$$A_1(a) \cos \psi = a B_1(a) \sin \psi$$

# Example (Van der Pol equation)

$$\ddot{x} + \omega^2 x = \mu f(x, \dot{x}) = \mu(1 - x^2)\dot{x}$$

As known, the system has an exponentially orbitally stable periodic solution $x_*(t)$ for $\mu > 0$. Let us try to approximate it as

$$x_*(t) = a(t) \cos \psi(t), \quad \dot{a}(t) = \mu A_1(a(t)), \; \dot{\psi}(t) = \omega + \mu B_1(a(t))$$

where the functions $A_1(\cdot)$, $B_1(\cdot)$ are to be found.

---

Step 1: To get the first equation relating $A_1(\cdot)$, $B_1(\cdot)$, one can postulate the form of $\dot{x}_*$ as

$$\begin{aligned} \dot{x}_* &= -a\omega \sin \psi = \dot{a} \cos \psi - a \sin \psi \, \dot{\psi} \\ &= \mu A_1(a) \cos \psi - a\omega \sin \psi - \mu a B_1(a) \sin \psi \end{aligned}$$

$$\Downarrow$$

$$A_1(a) \cos \psi = a B_1(a) \sin \psi$$

## Example (Van der Pol equation)

$$\ddot{x} + \omega^2 x = \mu f(x, \dot{x}) = \mu(1 - x^2)\dot{x}$$

As known, the system has an exponentially orbitally stable periodic solution $x_*(t)$ for $\mu > 0$. Let us try to approximate it as

$$x_*(t) = a(t)\cos\psi(t), \quad \dot{a}(t) = \mu A_1(a(t)), \ \dot{\psi}(t) = \omega + \mu B_1(a(t))$$

where the functions $A_1(\cdot)$, $B_1(\cdot)$ are to be found.

---

Step 1: To get the first equation relating $A_1(\cdot)$, $B_1(\cdot)$, one can postulate the form of $\dot{x}_*$ as

$$\dot{x}_* \quad = \quad -a\omega\sin\psi$$

---

Step 1: To get the first equation relating $A_1(\cdot)$, $B_1(\cdot)$, one can postulate the form of $\dot{x}_*$ as

$$\dot{x}_* \quad = \quad -a\omega\sin\psi = \dot{a}\cos\psi - a\sin\psi\,\dot{\psi}$$

## Example (Van der Pol equation)

$$\ddot{x} + \omega^2 x = \mu f(x, \dot{x}) = \mu(1 - x^2)\dot{x}$$

As known, the system has an exponentially orbitally stable periodic solution $x_*(t)$ for $\mu > 0$. Let us try to approximate it as

$$x_*(t) = a(t) \cos \psi(t), \quad \dot{a}(t) = \mu A_1(a(t)), \ \dot{\psi}(t) = \omega + \mu B_1(a(t))$$

where the functions $A_1(\cdot)$, $B_1(\cdot)$ are to be found.

---

Step 1: To get the first equation relating $A_1(\cdot)$, $B_1(\cdot)$, one can postulate the form of $\dot{x}_*$ as

$$\dot{x}_* \quad = \quad -a\omega \sin \psi \ = \ \dot{a} \cos \psi - a \sin \psi \, \dot{\psi}$$

Step 1: To get the first equation relating $A_1(\cdot)$, $B_1(\cdot)$, one can postulate the form of $\dot{x}_*$ as

$$\dot{x}_* \quad = \quad -a\omega \sin \psi \ = \ \dot{a} \cos \psi - a \sin \psi \, \dot{\psi}$$

## Example (Van der Pol equation)

$$\ddot{x} + \omega^2 x = \mu f(x, \dot{x}) = \mu(1 - x^2)\dot{x}$$

As known, the system has an exponentially orbitally stable periodic solution $x_*(t)$ for $\mu > 0$. Let us try to approximate it as

$$x_*(t) = a(t) \cos \psi(t), \quad \dot{a}(t) = \mu A_1(a(t)), \; \dot{\psi}(t) = \omega + \mu B_1(a(t))$$

where the functions $A_1(\cdot)$, $B_1(\cdot)$ are to be found.

---

Step 1: To get the first equation relating $A_1(\cdot)$, $B_1(\cdot)$, one can postulate the form of $\dot{x}_*$ as

$$\begin{aligned} \dot{x}_* &= -a\omega \sin \psi = \dot{a} \cos \psi - a \sin \psi \, \dot{\psi} \\ &= \mu A_1(a) \cos \psi - a\omega \sin \psi - \mu a B_1(a) \sin \psi \end{aligned}$$

$$\Downarrow$$

$$A_1(a) \cos \psi = a B_1(a) \sin \psi$$

## Example (Van der Pol equation)

$$\ddot{x} + \omega^2 x = \mu f(x, \dot{x}) = \mu(1 - x^2)\dot{x}$$

As known, the system has an exponentially orbitally stable periodic solution $x_*(t)$ for $\mu > 0$. Let us try to approximate it as

$$x_*(t) = a(t) \cos \psi(t), \quad \dot{a}(t) = \mu A_1(a(t)), \; \dot{\psi}(t) = \omega + \mu B_1(a(t))$$

where the functions $A_1(\cdot)$, $B_1(\cdot)$ are to be found.

---

Step 1: To get the first equation relating $A_1(\cdot)$, $B_1(\cdot)$, one can postulate the form of $\dot{x}_*$ as

$$\begin{aligned} \dot{x}_* &= -a\omega \sin \psi = \dot{a} \cos \psi - a \sin \psi \, \dot{\psi} \\ &= \mu A_1(a) \cos \psi - a\omega \sin \psi - \mu a B_1(a) \sin \psi \end{aligned}$$

$$\Downarrow$$

$$A_1(a) \cos \psi = a B_1(a) \sin \psi$$

# Example (Van der Pol equation)

Step 2: Substitute the approximation for $x_*$ into the equation:

$$\begin{aligned}\ddot{x}_* + \omega^2 x_* &= \tfrac{d}{dt}\Big[-a\omega\sin\psi\Big] + \omega^2\Big[a\cos\psi\Big]\\ &= -\dot{a}\omega\sin\psi - a\omega\cos\psi\dot{\psi} + \omega^2 a\cos\psi\\ &= -\mu A_1(a)\omega\sin\psi - a\omega\cos\psi\,[\omega+\mu B_1(a)] + \omega^2 a\cos\psi\\ &= -\mu A_1(a)\omega\sin\psi - \mu a\omega\cos\psi B_1(a) = \mu f\Big(a\cos\psi, -a\omega\sin\psi\Big)\end{aligned}$$

The equations from Steps 1-2 result in expressions:

$$\begin{aligned}A_1(a) &= -\frac{\sin\psi}{\omega} f\Big(a\cos\psi, -a\omega\sin\psi\Big)\\ B_1(a) &= -\frac{\cos\psi}{a\omega} f\Big(a\cos\psi, -a\omega\sin\psi\Big)\end{aligned}$$

Look at the right hand sides, they are dependent on $\psi$!

## Example (Van der Pol equation)

Step 2: Substitute the approximation for $x_*$ into the equation:

$$\begin{aligned}\ddot{x}_* + \omega^2 x_* &= \tfrac{d}{dt}\Big[-a\omega\sin\psi\Big] + \omega^2\Big[a\cos\psi\Big] \\ &= -\dot{a}\omega\sin\psi - a\omega\cos\psi\,\dot{\psi} + \omega^2 a\cos\psi \\ &= -\mu A_1(a)\omega\sin\psi - a\omega\cos\psi\,[\omega + \mu B_1(a)] + \omega^2 a\cos\psi \\ &= -\mu A_1(a)\omega\sin\psi - \mu a\omega\cos\psi\, B_1(a) = \mu f\Big(a\cos\psi, -a\omega\sin\psi\Big)\end{aligned}$$

---

The equations from Steps 1-2 result in expressions:

$$\begin{aligned}A_1(a) &= -\frac{\sin\psi}{\omega} f\Big(a\cos\psi, -a\omega\sin\psi\Big) \\ B_1(a) &= -\frac{\cos\psi}{a\omega} f\Big(a\cos\psi, -a\omega\sin\psi\Big)\end{aligned}$$

---

Look at the right hand sides, they are dependent on $\psi$!

## Example (Van der Pol equation)

Step 2: Substitute the approximation for $x_*$ into the equation:

$$\begin{aligned}\ddot{x}_* + \omega^2 x_* &= \tfrac{d}{dt}\Big[-a\omega\sin\psi\Big] + \omega^2\Big[a\cos\psi\Big] \\ &= -\dot{a}\omega\sin\psi - a\omega\cos\psi\dot{\psi} + \omega^2 a\cos\psi \\ &= -\mu A_1(a)\omega\sin\psi - a\omega\cos\psi\,[\omega + \mu B_1(a)] + \omega^2 a\cos\psi \\ &= -\mu A_1(a)\omega\sin\psi - \mu a\omega\cos\psi B_1(a) = \mu f\Big(a\cos\psi, -a\omega\sin\psi\Big)\end{aligned}$$

---

The equations from Steps 1-2 result in expressions:

$$\begin{aligned} A_1(a) &= -\frac{\sin\psi}{\omega} f\Big(a\cos\psi, -a\omega\sin\psi\Big) \\ B_1(a) &= -\frac{\cos\psi}{a\omega} f\Big(a\cos\psi, -a\omega\sin\psi\Big)\end{aligned}$$

---

Look at the right hand sides, they are dependent on $\psi$!

# Example (Van der Pol equation)

To remove such dependence, we average them by $\psi$ over period:

$$
\begin{aligned}
A_1(a) &= -\frac{1}{2\pi}\int_0^{2\pi} \frac{\sin\psi}{\omega} f\left(a\cos\psi, -a\omega\sin\psi\right) d\psi \\
&= \frac{a}{2}\left(1-\frac{a^2}{4}\right) \\
B_1(a) &= -\frac{1}{2\pi}\int_0^{2\pi} \frac{\cos\psi}{a\omega} f\left(a\cos\psi, -a\omega\sin\psi\right) d\psi \;=\; 0
\end{aligned}
$$

So that the approximation for $x_*(t)$ is

$$x_*(t) = a(t)\cos\psi(t), \quad \dot{a} = \mu\frac{a}{2}\left(1-\frac{a^2}{4}\right), \quad \dot{\psi} = \omega + \mu\cdot 0$$

## Example (Van der Pol equation)

To remove such dependence, we average them by $\psi$ over period:

$$\begin{aligned} A_1(a) &= -\frac{1}{2\pi}\int_0^{2\pi}\frac{\sin\psi}{\omega} f\left(a\cos\psi, -a\omega\sin\psi\right)d\psi \\ &= \frac{a}{2}\left(1-\frac{a^2}{4}\right) \\ B_1(a) &= -\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos\psi}{a\omega} f\left(a\cos\psi, -a\omega\sin\psi\right)d\psi \;=\; 0 \end{aligned}$$

---

So that the approximation for $x_*(t)$ is

$$x_*(t) = a(t)\cos\psi(t), \quad \dot{a} = \mu\frac{a}{2}\left(1-\frac{a^2}{4}\right), \quad \dot{\psi} = \omega + \mu\cdot 0$$

# Example (Van der Pol equation)

The equation for the amplitude $a$ of the approximation can be further rewritten as

$$2a\dot{a} = \frac{d}{dt}\left[a^2\right] = \mu a^2 \left(1 - \frac{a^2}{4}\right)$$

It has equilibria at

$$a = 0, \quad a = 2$$

It is seen that

- the equilibrium at $a = 0$ is unstable;
- the equilibrium at $a = 2$ is stable

---

The final form of the improved 0-order approximation of the periodic solution $x_*(t)$ is

$$x_*(t) \approx a(t) \cos \psi(t) = 2 \cos \omega t$$

# Example (Van der Pol equation)

The equation for the amplitude $a$ of the approximation can be further rewritten as

$$2a\dot{a} = \frac{d}{dt}\left[a^2\right] = \mu a^2 \left(1 - \frac{a^2}{4}\right)$$

It has equilibria at

$$a = 0, \quad a = 2$$

It is seen that

- the equilibrium at $a = 0$ is unstable;
- the equilibrium at $a = 2$ is stable

---

The final form of the improved 0-order approximation of the periodic solution $x_*(t)$ is

$$x_*(t) \approx a(t)\cos\psi(t) = 2\cos\omega t$$

## Example (Van der Pol equation)

The equation for the amplitude $a$ of the approximation can be further rewritten as

$$2a\dot{a} = \frac{d}{dt}\left[a^2\right] = \mu a^2 \left(1 - \frac{a^2}{4}\right)$$

It has equilibria at

$$a = 0, \quad a = 2$$

It is seen that

- the equilibrium at $a = 0$ is unstable;
- the equilibrium at $a = 2$ is stable

---

The final form of the improved 0-order approximation of the periodic solution $x_*(t)$ is

$$x_*(t) \approx a(t)\cos\psi(t) = 2\cos\omega t$$

## Example (Van der Pol equation)

The equation for the amplitude $a$ of the approximation can be further rewritten as

$$2a\dot{a} = \frac{d}{dt}\left[a^2\right] = \mu a^2 \left(1 - \frac{a^2}{4}\right)$$

It has equilibria at

$$a = 0, \quad a = 2$$

It is seen that

- the equilibrium at $a = 0$ is unstable;
- the equilibrium at $a = 2$ is stable

---

The final form of the improved 0-order approximation of the periodic solution $x_*(t)$ is

$$x_*(t) \approx a(t) \cos \psi(t) = 2 \cos \omega t$$

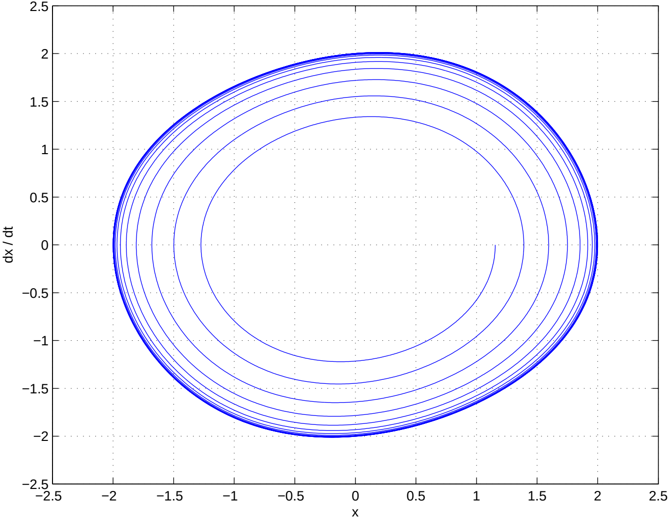


**Figure 3:** The simulated solution of the system when $\omega = 1$ and $\mu = 0.1$.

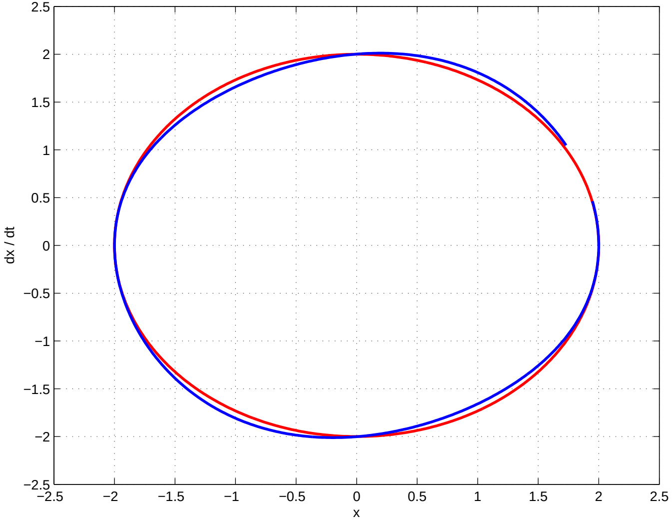


**Figure 4:** The simulated periodic solution $x_*(t)$ vs. its improved 0-order approximation, when $\omega = 1$ and $\mu = 0.1$