

Lecture 2

Constrained Mechanical Systems

Anton Shiriaev

Stockholm, 2018-09-07

Learning outcomes: Nonlinear mechanical systems with constraints. Classification of constraints. Stability of nonlinear mechanical systems with constraints. Examples.

1. Systems subject to holonomic constraints

1.1 Example: constrained point-mass dynamics

2. Systems with non-holonomic constraints

2.1 Example: constrained two point-masses dynamics

3. Stability of a motion of a mechanical system

3.1 Example: a mathematical pendulum

3.2 Example: a pendulum on a cart

3.3 Lagrange-Dirichlet theorem

3.4 Example: restricted 3 body problem

Systems subject to holonomic constraints

Example: point-mass dynamics in excessive coordinates

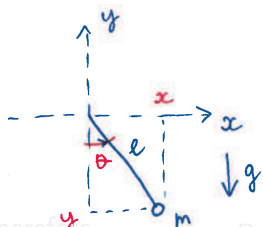
Let a point mass with coordinates (x, y) move in the vertical plane affected by the gravity. Assume that the distance of the point to the origin is kept constant

$$f(x(t), y(t)) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall t.$$

The dynamics of the point are

$$m \cdot \ddot{x} = R_x, \quad m \cdot \ddot{y} = R_y - m \cdot g,$$

where $R = [R_x; R_y]$ is the reaction force due to the constraint.



Example: point-mass dynamics in excessive coordinates

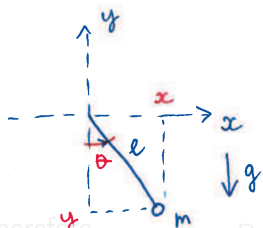
Let a point mass with coordinates (x, y) move in the vertical plane affected by the gravity. Assume that the distance of the point to the origin is kept constant

$$f(x(t), y(t)) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall t.$$

The dynamics of the point are

$$m \cdot \ddot{x} = R_x, \quad m \cdot \ddot{y} = R_y - m \cdot g,$$

where $R = [R_x; R_y]$ is the reaction force due to the constraint.



How to find the reaction force

$$R = R(x, y, \dot{x}, \dot{y})?$$

Example: point-mass dynamics in excessive coordinates

Let a point mass with coordinates (x, y) move in the vertical plane affected by the gravity. Assume that the distance of the point to the origin is kept constant

$$f(x(t), y(t)) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall t.$$

The dynamics of the point are

$$m \cdot \ddot{x} = R_x, \quad m \cdot \ddot{y} = R_y - m \cdot g,$$

where $R = [R_x; R_y]$ is the reaction force due to the constraint.

The constraint $f(\cdot) \equiv 0$ implies that

$$\frac{d}{dt}f = 2x(t) \cdot \dot{x}(t) + 2y(t) \cdot \dot{y}(t) \equiv 0, \quad \forall t.$$

Therefore

$$R_x = \lambda \cdot x, \quad R_y = \lambda \cdot y.$$

Example: point-mass dynamics in excessive coordinates

Let a point mass with coordinates (x, y) move in the vertical plane affected by the gravity. Assume that the distance of the point to the origin is kept constant

$$f(x(t), y(t)) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall t.$$

The dynamics of the point are

$$m \cdot \ddot{x} = R_x, \quad m \cdot \ddot{y} = R_y - m \cdot g,$$

where $R = [R_x; R_y]$ is the reaction force due to the constraint.

The constraint $f(\cdot) \equiv 0$ implies that

$$\frac{d}{dt}f = 2x(t) \cdot \dot{x}(t) + 2y(t) \cdot \dot{y}(t) \equiv 0, \quad \forall t.$$

The reaction force $R(\cdot)$ cannot change the energy of the system

$$R_x \cdot \dot{x}(t) + R_y \cdot \dot{y}(t) \equiv 0, \quad \forall t.$$

Example: point-mass dynamics in excessive coordinates

Let a point mass with coordinates (x, y) move in the vertical plane affected by the gravity. Assume that the distance of the point to the origin is kept constant

$$f(x(t), y(t)) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall t.$$

The dynamics of the point are

$$m \cdot \ddot{x} = R_x, \quad m \cdot \ddot{y} = R_y - m \cdot g,$$

where $R = [R_x; R_y]$ is the reaction force due to the constraint.

The constraint $f(\cdot) \equiv 0$ implies that

$$\frac{d}{dt}f = 2x(t) \cdot \dot{x}(t) + 2y(t) \cdot \dot{y}(t) \equiv 0, \quad \forall t.$$

The reaction force $R(\cdot)$ cannot change the energy of the system

$$R_x \cdot \dot{x}(t) + R_y \cdot \dot{y}(t) \equiv 0, \quad \forall t.$$

Example: point-mass dynamics in excessive coordinates

Let a point mass with coordinates (x, y) move in the vertical plane affected by the gravity. Assume that the distance of the point to the origin is kept constant

$$f(x(t), y(t)) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall t.$$

The dynamics of the point are

$$m \cdot \ddot{x} = R_x, \quad m \cdot \ddot{y} = R_y - m \cdot g,$$

where $R = [R_x; R_y]$ is the reaction force due to the constraint.

Therefore $R_x = \lambda \cdot x, \quad R_y = \lambda \cdot y.$

Example: point-mass dynamics in excessive coordinates

Let a point mass with coordinates (x, y) move in the vertical plane affected by the gravity. Assume that the distance of the point to the origin is kept constant

$$f(x(t), y(t)) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall t.$$

The dynamics of the point are

$$m \cdot \ddot{x} = R_x, \quad m \cdot \ddot{y} = R_y - m \cdot g,$$

where $R = [R_x; R_y]$ is the reaction force due to the constraint.

Therefore $R_x = \lambda \cdot x, \quad R_y = \lambda \cdot y.$

For defining λ consider the 2nd derivative of the constraint $f(\cdot) \equiv 0$

$$\begin{aligned} 0 \equiv \frac{d^2}{dt^2} f &= 2\dot{x}^2 + 2x \cdot \ddot{x} + 2\dot{y}^2 + 2y \cdot \ddot{y} \\ &= 2\dot{x}^2 + 2x \cdot \left(\frac{1}{m} \lambda \cdot x\right) + 2\dot{y}^2 + 2y \cdot \left(\frac{1}{m} \lambda \cdot y - g\right) \\ &= 2\dot{x}^2 + 2\dot{y}^2 + 2\frac{1}{m} \lambda \cdot (x^2 + y^2) - 2y \cdot g \end{aligned}$$

Example: point-mass dynamics in excessive coordinates

Let a point mass with coordinates (x, y) move in the vertical plane affected by the gravity. Assume that the distance of the point to the origin is kept constant

$$f(x(t), y(t)) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall t.$$

The dynamics of the point are

$$m \cdot \ddot{x} = R_x, \quad m \cdot \ddot{y} = R_y - m \cdot g,$$

where $R = [R_x; R_y]$ is the reaction force due to the constraint.

Therefore $R_x = \lambda \cdot x, \quad R_y = \lambda \cdot y.$

The Lagrangian multiplier is then equal to

$$\lambda = \frac{m}{l^2} (y \cdot g - \dot{x}^2 - \dot{y}^2)$$

Example: point-mass dynamics in excessive coordinates

Let a point mass with coordinates (x, y) move in the vertical plane affected by the gravity. Assume that the distance of the point to the origin is kept constant

$$f(x(t), y(t)) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall t.$$

The dynamics of the point are

$$m \cdot \ddot{x} = R_x, \quad m \cdot \ddot{y} = R_y - m \cdot g,$$

where $R = [R_x; R_y]$ is the reaction force due to the constraint.

Therefore $R_x = \lambda \cdot x, \quad R_y = \lambda \cdot y.$

The point mass dynamics in excessive coordinates (x, y) are

$$m \cdot \ddot{x} = \frac{m}{l^2} (y \cdot g - \dot{x}^2 - \dot{y}^2) \cdot x$$

$$m \cdot \ddot{y} = \frac{m}{l^2} (y \cdot g - \dot{x}^2 - \dot{y}^2) \cdot y - m \cdot g$$

Example: point-mass dynamics in generalized coordinates

To derive the point-mass dynamics with the constraint

$$f(x(t), y(t)) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall t$$

observe that the point's position is determined by the angle θ as

$$x(t) = l \cdot \sin \theta(t), \quad y(t) = -l \cdot \cos \theta(t).$$

The Lagrangian of the system is then

$$\begin{aligned} \mathcal{L} = K - \Pi &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - m \cdot g \cdot y \\ &= \frac{1}{2} m \cdot l^2 \cdot \dot{\theta}^2 + m \cdot g \cdot l \cdot \cos \theta \end{aligned}$$

The dynamics are

$$\begin{aligned} 0 &= \frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right] - \frac{\partial \mathcal{L}}{\partial \theta} = m \cdot l^2 \cdot \ddot{\theta} + m \cdot g \cdot l \cdot \sin \theta \\ &= m \cdot l^2 \cdot \left(\ddot{\theta} + \frac{g}{l} \cdot \sin \theta \right), \end{aligned}$$

which is the equation of the mathematical pendulum.

Example: point-mass dynamics in generalized coordinates

To derive the point-mass dynamics with the constraint

$$f(x(t), y(t)) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall t$$

observe that the point's position is determined by the angle θ as

$$x(t) = l \cdot \sin \theta(t), \quad y(t) = -l \cdot \cos \theta(t).$$

The Lagrangian of the system is then

$$\begin{aligned} \mathcal{L} = K - \Pi &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - m \cdot g \cdot y \\ &= \frac{1}{2} m \cdot l^2 \cdot \dot{\theta}^2 + m \cdot g \cdot l \cdot \cos \theta \end{aligned}$$

The dynamics are

$$\begin{aligned} 0 &= \frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right] - \frac{\partial \mathcal{L}}{\partial \theta} = m \cdot l^2 \cdot \ddot{\theta} + m \cdot g \cdot l \cdot \sin \theta \\ &= m \cdot l^2 \cdot \left(\ddot{\theta} + \frac{g}{l} \cdot \sin \theta \right), \end{aligned}$$

which is the equation of the mathematical pendulum.

Example: point-mass dynamics in generalized coordinates

To derive the point-mass dynamics with the constraint

$$f(x(t), y(t)) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall t$$

observe that the point's position is determined by the angle θ as

$$x(t) = l \cdot \sin \theta(t), \quad y(t) = -l \cdot \cos \theta(t).$$

The Lagrangian of the system is then

$$\begin{aligned} \mathcal{L} = K - \Pi &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - m \cdot g \cdot y \\ &= \frac{1}{2} m \cdot l^2 \cdot \dot{\theta}^2 + m \cdot g \cdot l \cdot \cos \theta \end{aligned}$$

The dynamics are

$$\begin{aligned} 0 &= \frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right] - \frac{\partial \mathcal{L}}{\partial \theta} = m \cdot l^2 \cdot \ddot{\theta} + m \cdot g \cdot l \cdot \sin \theta \\ &= m \cdot l^2 \cdot \left(\ddot{\theta} + \frac{g}{l} \cdot \sin \theta \right), \end{aligned}$$

which is the equation of the mathematical pendulum.

Example: point-mass dynamics

The equations written in excessive coordinates (x, y)

$$\begin{aligned}m \cdot \ddot{x} &= \frac{m}{l^2} (y \cdot g - \dot{x}^2 - \dot{y}^2) \cdot x \\m \cdot \ddot{y} &= \frac{m}{l^2} (y \cdot g - \dot{x}^2 - \dot{y}^2) \cdot y - m \cdot g\end{aligned}\tag{1}$$

and the equation written in generalized coordinate θ

$$\ddot{\theta} + \frac{g}{l} \cdot \sin \theta = 0\tag{2}$$

represent the dynamics of the same system provided that the initial conditions of both differential equations are appropriately chosen.

However, for mechanical systems with constraints

- the equations of the form (1) can be always derived,
- while the equations of the form (2) might not.

Example: point-mass dynamics

The equations written in excessive coordinates (x, y)

$$\begin{aligned}m \cdot \ddot{x} &= \frac{m}{l^2} (y \cdot g - \dot{x}^2 - \dot{y}^2) \cdot x \\m \cdot \ddot{y} &= \frac{m}{l^2} (y \cdot g - \dot{x}^2 - \dot{y}^2) \cdot y - m \cdot g\end{aligned}\tag{1}$$

and the equation written in generalized coordinate θ

$$\ddot{\theta} + \frac{g}{l} \cdot \sin \theta = 0\tag{2}$$

represent the dynamics of the same system provided that the initial conditions of both differential equations are appropriately chosen.

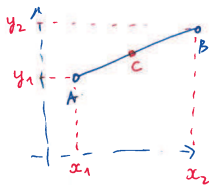
However, for mechanical systems with constraints

- the equations of the form (1) can be always derived,
- while the equations of the form (2) might not.

Systems with non-holonomic constraints

Example: constrained two point-masses dynamics

Consider two point masses of $m = 1$ [kg] each connected by massless rod of length l and moving in the vertical plane.



Constraint No. 1: $(x_1(t) - x_2(t))^2 + (y_1(t) - y_2(t))^2 = l^2, \forall t$

Constraint No. 2: Assume that the velocity of the center of the rod – point C on the plot

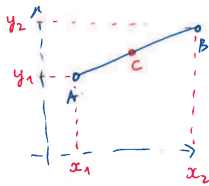
$$\vec{v}_C = \left[\frac{1}{2}(\dot{x}_1 + \dot{x}_2); \frac{1}{2}(\dot{y}_1 + \dot{y}_2) \right]$$

always aligned with the rod.

$$(x_2(t) - x_1(t)) (\dot{y}_1(t) + \dot{y}_2(t)) - (y_2(t) - y_1(t)) (\dot{x}_1(t) + \dot{x}_2(t)) \equiv 0$$

Example: constrained two point-masses dynamics

Consider two point masses of $m = 1$ [kg] each connected by massless rod of length l and moving in the vertical plane.



Constraint No. 1: $(x_1(t) - x_2(t))^2 + (y_1(t) - y_2(t))^2 = l^2, \forall t$

Constraint No. 2: Assume that the velocity of the center of the rod – point C on the plot

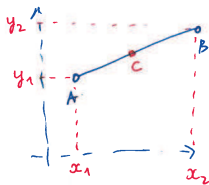
$$\vec{v}_C = \left[\frac{1}{2}(\dot{x}_1 + \dot{x}_2); \frac{1}{2}(\dot{y}_1 + \dot{y}_2) \right]$$

always aligned with the rod.

$$(x_2(t) - x_1(t)) (\dot{y}_1(t) + \dot{y}_2(t)) - (y_2(t) - y_1(t)) (\dot{x}_1(t) + \dot{x}_2(t)) \equiv 0$$

Example: constrained two point-masses dynamics

Consider two point masses of $m = 1$ [kg] each connected by massless rod of length l and moving in the vertical plane.



Constraint No. 1: $(x_1(t) - x_2(t))^2 + (y_1(t) - y_2(t))^2 = l^2, \forall t$

Constraint No. 2: Assume that the velocity of the center of the rod – point **C** on the plot

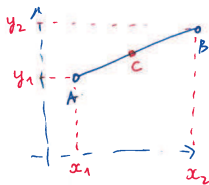
$$\vec{v}_C = \left[\frac{1}{2}(\dot{x}_1 + \dot{x}_2); \frac{1}{2}(\dot{y}_1 + \dot{y}_2) \right]$$

always aligned with the rod.

$$(x_2(t) - x_1(t)) (\dot{y}_1(t) + \dot{y}_2(t)) - (y_2(t) - y_1(t)) (\dot{x}_1(t) + \dot{x}_2(t)) \equiv 0$$

Example: constrained two point-masses dynamics

Consider two point masses of $m = 1$ [kg] each connected by massless rod of length l and moving in the vertical plane.



Constraint No. 1: $(x_1(t) - x_2(t))^2 + (y_1(t) - y_2(t))^2 = l^2, \forall t$

Constraint No. 2: Assume that the velocity of the center of the rod – point C on the plot

$$\vec{v}_C = \left[\frac{1}{2}(\dot{x}_1 + \dot{x}_2); \frac{1}{2}(\dot{y}_1 + \dot{y}_2) \right]$$

always aligned with the rod.

$$\left(x_2(t) - x_1(t) \right) \left(\dot{y}_1(t) + \dot{y}_2(t) \right) - \left(y_2(t) - y_1(t) \right) \left(\dot{x}_1(t) + \dot{x}_2(t) \right) \equiv 0$$

Example: constrained two point-masses dynamics

The dynamics of the system with coordinates (x_1, y_1, x_2, y_2) are

$$\begin{aligned}\ddot{x}_1 &= R_{x_1}^{(1)} + R_{x_1}^{(2)} & \ddot{x}_2 &= R_{x_2}^{(1)} + R_{x_2}^{(2)} \\ \ddot{y}_1 &= R_{y_1}^{(1)} + R_{y_1}^{(2)} - g & \ddot{y}_2 &= R_{y_2}^{(1)} + R_{y_2}^{(2)} - g\end{aligned}$$

Here $R^{(1)}$ and $R^{(2)}$ are the reaction forces due to constraints

$$R^{(1)} = [R_{x_1}^{(1)}; R_{y_1}^{(1)}; R_{x_2}^{(1)}; R_{y_2}^{(1)}], \quad R^{(2)} = [R_{x_1}^{(2)}; R_{y_1}^{(2)}; R_{x_2}^{(2)}; R_{y_2}^{(2)}]$$

Components of the reaction forces are determined from the assumption that such forces do not dissipate or increase the energy of the system along its motions

$$R_{x_1}^{(i)} \cdot \dot{x}_1 + R_{y_1}^{(i)} \cdot \dot{y}_1 + R_{x_2}^{(i)} \cdot \dot{x}_2 + R_{y_2}^{(i)} \cdot \dot{y}_2 \equiv 0, \quad i = 1, 2.$$

Example: constrained two point-masses dynamics

The dynamics of the system with coordinates (x_1, y_1, x_2, y_2) are

$$\begin{aligned}\ddot{x}_1 &= R_{x_1}^{(1)} + R_{x_1}^{(2)} & \ddot{x}_2 &= R_{x_2}^{(1)} + R_{x_2}^{(2)} \\ \ddot{y}_1 &= R_{y_1}^{(1)} + R_{y_1}^{(2)} - g & \ddot{y}_2 &= R_{y_2}^{(1)} + R_{y_2}^{(2)} - g\end{aligned}$$

Here $R^{(1)}$ and $R^{(2)}$ are the reaction forces due to constraints

$$R^{(1)} = [R_{x_1}^{(1)}; R_{y_1}^{(1)}; R_{x_2}^{(1)}; R_{y_2}^{(1)}], \quad R^{(2)} = [R_{x_1}^{(2)}; R_{y_1}^{(2)}; R_{x_2}^{(2)}; R_{y_2}^{(2)}]$$

Components of the reaction forces are determined from the assumption that such forces do not dissipate or increase the energy of the system along its motions

$$R_{x_1}^{(i)} \cdot \dot{x}_1 + R_{y_1}^{(i)} \cdot \dot{y}_1 + R_{x_2}^{(i)} \cdot \dot{x}_2 + R_{y_2}^{(i)} \cdot \dot{y}_2 \equiv 0, \quad i = 1, 2.$$

Example: constrained two point-masses dynamics

The dynamics of the system with coordinates (x_1, y_1, x_2, y_2) are

$$\begin{aligned}\ddot{x}_1 &= R_{x_1}^{(1)} + R_{x_1}^{(2)} & \ddot{x}_2 &= R_{x_2}^{(1)} + R_{x_2}^{(2)} \\ \ddot{y}_1 &= R_{y_1}^{(1)} + R_{y_1}^{(2)} - g & \ddot{y}_2 &= R_{y_2}^{(1)} + R_{y_2}^{(2)} - g\end{aligned}$$

Here $R^{(1)}$ and $R^{(2)}$ are the reaction forces due to constraints

$$R^{(1)} = [R_{x_1}^{(1)}; R_{y_1}^{(1)}; R_{x_2}^{(1)}; R_{y_2}^{(1)}], \quad R^{(2)} = [R_{x_1}^{(2)}; R_{y_1}^{(2)}; R_{x_2}^{(2)}; R_{y_2}^{(2)}]$$

Components of the reaction forces are determined from the assumption that such forces do not dissipate or increase the energy of the system along its motions

$$R_{x_1}^{(i)} \cdot \dot{x}_1 + R_{y_1}^{(i)} \cdot \dot{y}_1 + R_{x_2}^{(i)} \cdot \dot{x}_2 + R_{y_2}^{(i)} \cdot \dot{y}_2 \equiv 0, \quad i = 1, 2.$$

Observation: The system dynamics are written in 4 excessive coordinates and have 2 constraints. But one cannot reduce a number of coordinates to 2 and derive the dynamics! ☹️☹️☹️

Example: constrained two point-masses dynamics

The dynamics of the system with coordinates (x_1, y_1, x_2, y_2) are

$$\begin{aligned}\ddot{x}_1 &= R_{x_1}^{(1)} + R_{x_1}^{(2)} & \ddot{x}_2 &= R_{x_2}^{(1)} + R_{x_2}^{(2)} \\ \ddot{y}_1 &= R_{y_1}^{(1)} + R_{y_1}^{(2)} - g & \ddot{y}_2 &= R_{y_2}^{(1)} + R_{y_2}^{(2)} - g\end{aligned}$$

Here $R^{(1)}$ and $R^{(2)}$ are the reaction forces due to constraints

$$R^{(1)} = [R_{x_1}^{(1)}; R_{y_1}^{(1)}; R_{x_2}^{(1)}; R_{y_2}^{(1)}], \quad R^{(2)} = [R_{x_1}^{(2)}; R_{y_1}^{(2)}; R_{x_2}^{(2)}; R_{y_2}^{(2)}]$$

Components of the reaction forces are determined from the assumption that such forces do not dissipate or increase the energy of the system along its motions

$$R_{x_1}^{(i)} \cdot \dot{x}_1 + R_{y_1}^{(i)} \cdot \dot{y}_1 + R_{x_2}^{(i)} \cdot \dot{x}_2 + R_{y_2}^{(i)} \cdot \dot{y}_2 \equiv 0, \quad i = 1, 2.$$

Observation: The system dynamics are integrable in the sense that all the solutions can be found explicitly! ☺☺☺ see the homework!

Stability of a motion of a mechanical system

Example: a mathematical pendulum

Let us investigate a stability of equilibriums of the system

$$\ddot{\theta} + \frac{g}{l} \cdot \sin \theta = 0. \quad (3)$$

There are two equilibriums (mod 2π): $\theta_e = 0$ and $\theta_e = \pi$.

Linearization of the dynamics at $\theta_e = \pi$ results in

$$\ddot{z} - \frac{g}{l} \cdot z = 0.$$

The equilibrium of this linear system is unstable. Therefore, the equilibrium $\theta_e = \pi$ of the nonlinear system (3) is unstable as well.

Linearization of the dynamics at $\theta_e = 0$ results in

$$\ddot{z} + \frac{g}{l} \cdot z = 0.$$

This linear system has the center at the origin and the nonlinear dynamics (3) has the first integral. Therefore, the nonlinear system (3) has the center at $\theta_e = 0$ and this equilibrium is stable!

Example: a mathematical pendulum

Let us investigate a stability of equilibriums of the system

$$\ddot{\theta} + \frac{g}{l} \cdot \sin \theta = 0. \quad (3)$$

There are two equilibriums (mod 2π): $\theta_e = 0$ and $\theta_e = \pi$.

Linearization of the dynamics at $\theta_e = \pi$ results in

$$\ddot{z} - \frac{g}{l} \cdot z = 0.$$

The equilibrium of this linear system is unstable. Therefore, the equilibrium $\theta_e = \pi$ of the nonlinear system (3) is unstable as well.

Linearization of the dynamics at $\theta_e = 0$ results in

$$\ddot{z} + \frac{g}{l} \cdot z = 0.$$

This linear system has the center at the origin and the nonlinear dynamics (3) has the first integral. Therefore, the nonlinear system (3) has the center at $\theta_e = 0$ and this equilibrium is stable!

Example: a mathematical pendulum

Let us investigate a stability of equilibriums of the system

$$\ddot{\theta} + \frac{g}{l} \cdot \sin \theta = 0. \quad (3)$$

There are two equilibriums (mod 2π): $\theta_e = 0$ and $\theta_e = \pi$.

Linearization of the dynamics at $\theta_e = \pi$ results in

$$\ddot{z} - \frac{g}{l} \cdot z = 0.$$

The equilibrium of this linear system is unstable. Therefore, the equilibrium $\theta_e = \pi$ of the nonlinear system (3) is unstable as well.

Linearization of the dynamics at $\theta_e = 0$ results in

$$\ddot{z} + \frac{g}{l} \cdot z = 0.$$

This linear system has the center at the origin and the nonlinear dynamics (3) has the first integral. Therefore, the nonlinear system (3) has the center at $\theta_e = 0$ and this equilibrium is stable!

Example: a mathematical pendulum

Let us investigate a stability of equilibriums of the system

$$\ddot{\theta} + \frac{g}{l} \cdot \sin \theta = 0. \quad (3)$$

There are two equilibriums (mod 2π): $\theta_e = 0$ and $\theta_e = \pi$.

Linearization of the dynamics at $\theta_e = \pi$ results in

$$\ddot{z} - \frac{g}{l} \cdot z = 0.$$

The equilibrium of this linear system is unstable. Therefore, the equilibrium $\theta_e = \pi$ of the nonlinear system (3) is unstable as well.

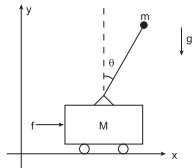
Linearization of the dynamics at $\theta_e = 0$ results in

$$\ddot{z} + \frac{g}{l} \cdot z = 0.$$

This linear system has the center at the origin and the nonlinear dynamics (3) has the first integral. Therefore, the nonlinear system (3) has the center at $\theta_e = 0$ and this equilibrium is stable!

Example: a pendulum on a cart

Consider a pendulum (a point of a mass m at the distance l from the suspension point) attached to a cart of a mass M , which freely moves on the horizontal with $f = 0$.



When $M = m = l = 1$ the dynamics in coordinates (x, θ) are

$$2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 = 0 (= f)$$

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

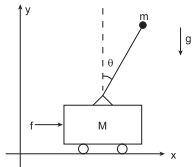
and have two sets of equilibriums:

- the pendulum is up, the cart is in any position: $\theta_e = 0, \forall x_e$
- the pendulum is down, the cart is in any position: $\theta_e = \pi, \forall x_e$

Let us investigate their stability!

Example: a pendulum on a cart

Consider a pendulum (a point of a mass m at the distance l from the suspension point) attached to a cart of a mass M , which freely moves on the horizontal with $f = 0$.



When $M = m = l = 1$ the dynamics in coordinates (x, θ) are

$$2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 = 0 (= f)$$

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

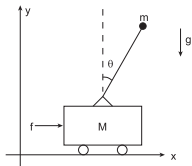
and have two sets of equilibriums:

- the pendulum is up, the cart is in any position: $\theta_e = 0, \forall x_e$
- the pendulum is down, the cart is in any position: $\theta_e = \pi, \forall x_e$

Let us investigate their stability!

Example: a pendulum on a cart

Consider a pendulum (a point of a mass m at the distance l from the suspension point) attached to a cart of a mass M , which freely moves on the horizontal with $f = 0$.



When $M = m = l = 1$ the dynamics in coordinates (x, θ) are

$$2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 = 0 (= f)$$

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

and have two sets of equilibriums:

- the pendulum is up, the cart is in any position: $\theta_e = 0, \forall x_e$
- the pendulum is down, the cart is in any position: $\theta_e = \pi, \forall x_e$

Let us investigate their stability!

Example: a pendulum on a cart

The system $2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 = 0$

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

has two integrals of motion (CoM): the total energy

$$E = \frac{1}{2} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}^T \begin{bmatrix} 2 & \cos \theta \\ \cos \theta & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + g \cdot \cos \theta$$

and the momentum conjugated to the x -coordinate

$$P(x(t), \theta(t), \dot{x}(t), \dot{\theta}(t)) = 2 \cdot \dot{x}(t) + \cos \theta(t) \cdot \dot{\theta}(t)$$

The last relation can be integrated again and leads to

$$\begin{aligned} x(t) &= x(0) + \frac{1}{2} \sin \theta(0) - \frac{1}{2} \sin \theta(t) + \frac{1}{2} P(x(0), \theta(0), \dot{x}(0), \dot{\theta}(0)) \cdot t \\ &= -\frac{1}{2} \sin \theta(t) + C_0 + C_1 \cdot t \end{aligned}$$

If one is able to compute $\theta(\cdot)$, then the formula gives $x(\cdot)$!

Example: a pendulum on a cart

The system $2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 = 0$

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

has two integrals of motion (CoM): the total energy

$$E = \frac{1}{2} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}^T \begin{bmatrix} 2 & \cos \theta \\ \cos \theta & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + g \cdot \cos \theta$$

and the momentum conjugated to the x -coordinate

$$P(x(t), \theta(t), \dot{x}(t), \dot{\theta}(t)) = 2 \cdot \dot{x}(t) + \cos \theta(t) \cdot \dot{\theta}(t)$$

The last relation can be integrated again and leads to

$$\begin{aligned} x(t) &= x(0) + \frac{1}{2} \sin \theta(0) - \frac{1}{2} \sin \theta(t) + \frac{1}{2} P(x(0), \theta(0), \dot{x}(0), \dot{\theta}(0)) \cdot t \\ &= -\frac{1}{2} \sin \theta(t) + C_0 + C_1 \cdot t \end{aligned}$$

If one is able to compute $\theta(\cdot)$, then the formula gives $x(\cdot)$!

Example: a pendulum on a cart

The system $2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 = 0$

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

has two integrals of motion (CoM): the total energy

$$E = \frac{1}{2} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}^T \begin{bmatrix} 2 & \cos \theta \\ \cos \theta & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + g \cdot \cos \theta$$

and the momentum conjugated to the x -coordinate

$$P(x(t), \theta(t), \dot{x}(t), \dot{\theta}(t)) = 2 \cdot \dot{x}(t) + \cos \theta(t) \cdot \dot{\theta}(t)$$

The last relation can be integrated again and leads to

$$\begin{aligned} x(t) &= x(0) + \frac{1}{2} \sin \theta(0) - \frac{1}{2} \sin \theta(t) + \frac{1}{2} P(x(0), \theta(0), \dot{x}(0), \dot{\theta}(0)) \cdot t \\ &= -\frac{1}{2} \sin \theta(t) + C_0 + C_1 \cdot t \end{aligned}$$

If one is able to compute $\theta(\cdot)$, then the formula gives $x(\cdot)$!

Example: a pendulum on a cart

For decoupling dynamics of the θ -variable in the system

$$2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 = 0$$

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

one can use the second equation.

Indeed

$$\cos \theta \cdot \overbrace{\frac{1}{2} \left(\sin \theta \cdot \dot{\theta}^2 - \cos \theta \cdot \ddot{\theta} \right)} = \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

substituting and collecting the terms one derives the equation

$$\left(1 - \frac{1}{2} \cos^2 \theta\right) \cdot \ddot{\theta} + \frac{1}{2} \cos \theta \cdot \sin \theta \cdot \dot{\theta}^2 - g \cdot \sin \theta = 0$$

What can one say on the dynamics of this system
in a vicinity of its equilibriums at 0 and at π ?

Example: a pendulum on a cart

For decoupling dynamics of the θ -variable in the system

$$2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 = 0$$

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

one can use the second equation.

Indeed

$$\cos \theta \cdot \frac{1}{2} \left(\overbrace{\sin \theta \cdot \dot{\theta}^2 - \cos \theta \cdot \ddot{\theta}} = \ddot{x} \right) + \ddot{\theta} - g \cdot \sin \theta = 0$$

substituting and collecting the terms one derives the equation

$$\left(1 - \frac{1}{2} \cos^2 \theta\right) \cdot \ddot{\theta} + \frac{1}{2} \cos \theta \cdot \sin \theta \cdot \dot{\theta}^2 - g \cdot \sin \theta = 0$$

What can one say on the dynamics of this system
in a vicinity of its equilibriums at 0 and at π ?

Example: a pendulum on a cart

For decoupling dynamics of the θ -variable in the system

$$2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 = 0$$

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

one can use the second equation.

Indeed

$$\cos \theta \cdot \frac{1}{2} \left(\overbrace{\sin \theta \cdot \dot{\theta}^2 - \cos \theta \cdot \ddot{\theta}} = \ddot{x} \right) + \ddot{\theta} - g \cdot \sin \theta = 0$$

substituting and collecting the terms one derives the equation

$$\left(1 - \frac{1}{2} \cos^2 \theta\right) \cdot \ddot{\theta} + \frac{1}{2} \cos \theta \cdot \sin \theta \cdot \dot{\theta}^2 - g \cdot \sin \theta = 0$$

What can one say on the dynamics of this system
in a vicinity of its equilibriums at 0 and at π ?

Example: a pendulum on a cart

The linearization of the system

$$\left(1 - \frac{1}{2} \cos^2 \theta\right) \cdot \ddot{\theta} + \frac{1}{2} \cos \theta \cdot \sin \theta \cdot \dot{\theta}^2 - g \cdot \sin \theta = 0$$

in a vicinity of the equilibrium $\theta_e = 0$ is unstable

$$\ddot{z} + \left[\frac{d}{d\theta} \left(\frac{-g \cdot \sin \theta}{1 - \frac{1}{2} \cos^2 \theta} \right) \right] \Big|_{\theta=0} \cdot z = \ddot{z} - 2 \cdot g \cdot z = 0$$

Its linearization at $\theta_e = \pi$ is stable

$$\ddot{z} + \left[\frac{d}{d\theta} \left(\frac{-g \cdot \sin \theta}{1 - \frac{1}{2} \cos^2 \theta} \right) \right] \Big|_{\theta=\pi} \cdot z = \ddot{z} + 2 \cdot g \cdot z = 0$$

In addition the system has the first integral

$$E_{red} = \frac{1}{2} \left(1 - \frac{1}{2} \cos^2 \theta\right) \cdot \dot{\theta}^2 + g \cdot \cos \theta$$

The nonlinear system has the center at $\theta_e = \pi$!

Example: a pendulum on a cart

The linearization of the system

$$\left(1 - \frac{1}{2} \cos^2 \theta\right) \cdot \ddot{\theta} + \frac{1}{2} \cos \theta \cdot \sin \theta \cdot \dot{\theta}^2 - g \cdot \sin \theta = 0$$

in a vicinity of the equilibrium $\theta_e = 0$ is unstable

$$\ddot{z} + \left[\frac{d}{d\theta} \left(\frac{-g \cdot \sin \theta}{1 - \frac{1}{2} \cos^2 \theta} \right) \right] \Big|_{\theta=0} \cdot z = \ddot{z} - 2 \cdot g \cdot z = 0$$

Its linearization at $\theta_e = \pi$ is stable

$$\ddot{z} + \left[\frac{d}{d\theta} \left(\frac{-g \cdot \sin \theta}{1 - \frac{1}{2} \cos^2 \theta} \right) \right] \Big|_{\theta=\pi} \cdot z = \ddot{z} + 2 \cdot g \cdot z = 0$$

In addition the system has the first integral

$$E_{red} = \frac{1}{2} \left(1 - \frac{1}{2} \cos^2 \theta\right) \cdot \dot{\theta}^2 + g \cdot \cos \theta$$

The nonlinear system has the center at $\theta_e = \pi$!

Example: a pendulum on a cart

The linearization of the system

$$\left(1 - \frac{1}{2} \cos^2 \theta\right) \cdot \ddot{\theta} + \frac{1}{2} \cos \theta \cdot \sin \theta \cdot \dot{\theta}^2 - g \cdot \sin \theta = 0$$

in a vicinity of the equilibrium $\theta_e = 0$ is unstable

$$\ddot{z} + \left[\frac{d}{d\theta} \left(\frac{-g \cdot \sin \theta}{1 - \frac{1}{2} \cos^2 \theta} \right) \right] \Big|_{\theta=0} \cdot z = \ddot{z} - 2 \cdot g \cdot z = 0$$

Its linearization at $\theta_e = \pi$ is stable

$$\ddot{z} + \left[\frac{d}{d\theta} \left(\frac{-g \cdot \sin \theta}{1 - \frac{1}{2} \cos^2 \theta} \right) \right] \Big|_{\theta=\pi} \cdot z = \ddot{z} + 2 \cdot g \cdot z = 0$$

In addition the system has the first integral

$$E_{red} = \frac{1}{2} \left(1 - \frac{1}{2} \cos^2 \theta\right) \cdot \dot{\theta}^2 + g \cdot \cos \theta$$

The nonlinear system has the center at $\theta_e = \pi$!

Example: a pendulum on a cart

To conclude:

- Any cart pendulum solution can be written as

$x(t) = \frac{1}{2} \sin \theta(t) + C_0 + C_1 t$ and $\theta(t)$ is a solution of reduced dynamics

- In a vicinity of the upright equilibrium $[\theta_e = 0, x = x_e]$ the reduced dynamics is hyperbolic, therefore **any of upright equilibriums is unstable**.
- In a vicinity of the downward equilibrium $[\theta_e = \pi, x = x_e]$ the reduced dynamics is stable, but $x(t)$ will drift with $C_1 \neq 0$. Hence **any of downward equilibriums is unstable as well**.

Theorem (1788)

If at the position of an isolated equilibrium of a conservative mechanical system with holonomic constraints the potential energy Π has a strict minimum, then this equilibrium is stable.

Example: restricted 3 body problem

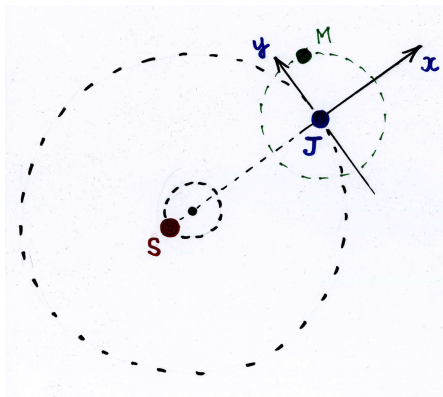
Equations of motion for the position of the Moon in rotating coordinate frame are

$$\begin{cases} \ddot{x} - 2m\dot{y} = \frac{\partial}{\partial x} F \\ \ddot{y} + 2m\dot{x} = \frac{\partial}{\partial y} F \end{cases}$$

Here

$$F = \frac{\kappa}{\sqrt{x^2 + y^2}} + \frac{3}{2}m^2x^2$$

m, κ are positive constants.



The system has the invariant: $I = \dot{x}^2 + \dot{y}^2 - 2F(x, y) + C$

Task: Analyze the dynamics in a vicinity of the periodic motion

Example: restricted 3 body problem

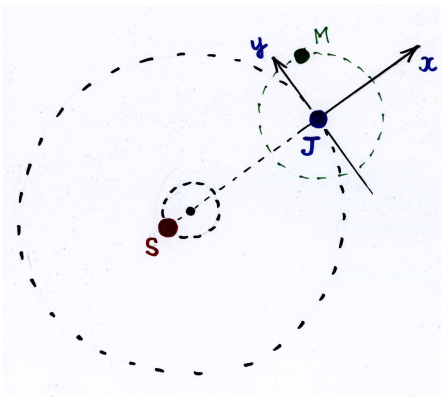
Equations of motion for the position of the Moon in rotating coordinate frame are

$$\begin{cases} \ddot{x} - 2m\dot{y} = \frac{\partial}{\partial x} F \\ \ddot{y} + 2m\dot{x} = \frac{\partial}{\partial y} F \end{cases}$$

Here

$$F = \frac{\kappa}{\sqrt{x^2 + y^2}} + \frac{3}{2}m^2x^2$$

m, κ are positive constants.



The system has the invariant: $I = \dot{x}^2 + \dot{y}^2 - 2F(x, y) + C$

Task: Analyze the dynamics in a vicinity of the periodic motion

Example: restricted 3 body problem

Equations of motion for the position of the Moon in rotating coordinate frame are

$$\begin{cases} \ddot{x} - 2m\dot{y} = \frac{\partial}{\partial x} F \\ \ddot{y} + 2m\dot{x} = \frac{\partial}{\partial y} F \end{cases}$$

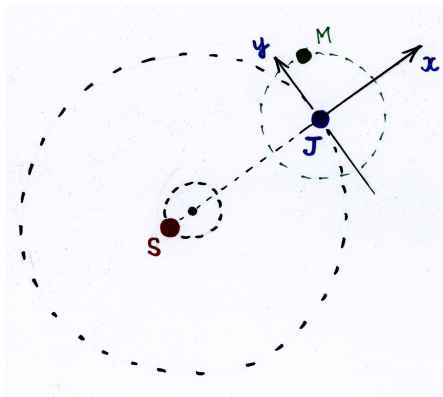
Here

$$F = \frac{\kappa}{\sqrt{x^2 + y^2}} + \frac{3}{2}m^2x^2$$

m, κ are positive constants.

The system has the invariant: $I = \dot{x}^2 + \dot{y}^2 - 2F(x, y) + C$

Task: Analyze the dynamics in a vicinity of the periodic motion



Example: elements of theory of G.W. Hill

Denote as $[x_p(t), y_p(t)]$ the periodic solution

Perturbed solutions $[x_p(t) + \delta x(t), y_p(t) + \delta y(t)]$ defined by

$$\begin{aligned} \frac{d^2}{dt^2} [\delta x] - 2m \frac{d}{dt} [\delta y] &= \\ &= \left[\frac{\partial^2}{\partial x^2} F(x_p(t), y_p(t)) \right] \delta x + \left[\frac{\partial^2}{\partial x \partial y} F(x_p(t), y_p(t)) \right] \delta y \end{aligned}$$

$$\begin{aligned} \frac{d^2}{dt^2} [\delta y] + 2m \frac{d}{dt} [\delta x] &= \\ &= \left[\frac{\partial^2}{\partial x \partial y} F(x_p(t), y_p(t)) \right] \delta x + \left[\frac{\partial^2}{\partial y^2} F(x_p(t), y_p(t)) \right] \delta y \end{aligned}$$

The integral Jacobi $I(\cdot)$ gives another relation

$$\begin{aligned} \frac{d}{dt} x_p(t) \frac{d}{dt} [\delta x] + \frac{d}{dt} y_p(t) \frac{d}{dt} [\delta y] &= \\ &= \left[\frac{\partial}{\partial x} F(x_p(t), y_p(t)) \right] \delta x + \left[\frac{\partial}{\partial y} F(x_p(t), y_p(t)) \right] \delta y \end{aligned}$$

Example: elements of theory of G.W. Hill

Denote as $[x_p(t), y_p(t)]$ the periodic solution

Perturbed solutions $[x_p(t) + \delta x(t), y_p(t) + \delta y(t)]$ defined by

$$\begin{aligned} \frac{d^2}{dt^2} [\delta x] - 2m \frac{d}{dt} [\delta y] &= \\ &= \left[\frac{\partial^2}{\partial x^2} F(x_p(t), y_p(t)) \right] \delta x + \left[\frac{\partial^2}{\partial x \partial y} F(x_p(t), y_p(t)) \right] \delta y \end{aligned}$$

$$\begin{aligned} \frac{d^2}{dt^2} [\delta y] + 2m \frac{d}{dt} [\delta x] &= \\ &= \left[\frac{\partial^2}{\partial x \partial y} F(x_p(t), y_p(t)) \right] \delta x + \left[\frac{\partial^2}{\partial y^2} F(x_p(t), y_p(t)) \right] \delta y \end{aligned}$$

The integral Jacobi $I(\cdot)$ gives another relation

$$\begin{aligned} \frac{d}{dt} x_p(t) \frac{d}{dt} [\delta x] + \frac{d}{dt} y_p(t) \frac{d}{dt} [\delta y] &= \\ &= \left[\frac{\partial}{\partial x} F(x_p(t), y_p(t)) \right] \delta x + \left[\frac{\partial}{\partial y} F(x_p(t), y_p(t)) \right] \delta y \end{aligned}$$

Example: elements of theory of G.W. Hill

Denote as $[x_p(t), y_p(t)]$ the periodic solution

Perturbed solutions $[x_p(t) + \delta x(t), y_p(t) + \delta y(t)]$ defined by

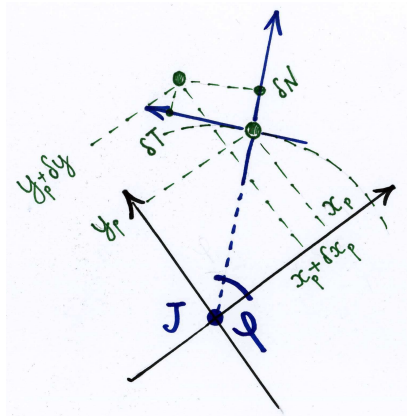
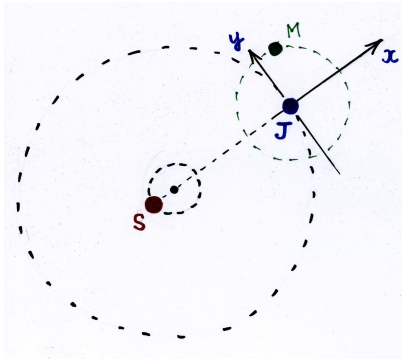
$$\begin{aligned} \frac{d^2}{dt^2} [\delta x] - 2m \frac{d}{dt} [\delta y] &= \\ &= \left[\frac{\partial^2}{\partial x^2} F(x_p(t), y_p(t)) \right] \delta x + \left[\frac{\partial^2}{\partial x \partial y} F(x_p(t), y_p(t)) \right] \delta y \end{aligned}$$

$$\begin{aligned} \frac{d^2}{dt^2} [\delta y] + 2m \frac{d}{dt} [\delta x] &= \\ &= \left[\frac{\partial^2}{\partial x \partial y} F(x_p(t), y_p(t)) \right] \delta x + \left[\frac{\partial^2}{\partial y^2} F(x_p(t), y_p(t)) \right] \delta y \end{aligned}$$

The integral Jacobi $I(\cdot)$ gives another relation

$$\begin{aligned} \frac{d}{dt} x_p(t) \frac{d}{dt} [\delta x] + \frac{d}{dt} y_p(t) \frac{d}{dt} [\delta y] &= \\ &= \left[\frac{\partial}{\partial x} F(x_p(t), y_p(t)) \right] \delta x + \left[\frac{\partial}{\partial y} F(x_p(t), y_p(t)) \right] \delta y \end{aligned}$$

Example: elements of theory of G.W. Hill



Transform of coordinates into normal (δN) and tangent (δT)

$$\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \delta T \\ \delta N \end{bmatrix}$$

Example: elements of theory of G.W. Hill

In a vicinity of the motion the original coordinates

$$\left[x, y, \dot{x}, \dot{y} \right]$$

are changed into

$$\left[\phi, I, N, \dot{N} \right]$$

The linearization of $\phi(\cdot)$ is not important: it perpetually rotates

The linearization of $I(\cdot)$ is straightforward: $\frac{d}{dt} [\delta I] \equiv 0$

The linearization of $[N, \dot{N}]$ is the famous Hill's equation

$$\frac{d^2}{dt^2} [\delta N] + \Phi(t) \delta N = 0$$

Example: elements of theory of G.W. Hill

In a vicinity of the motion the original coordinates

$$\left[x, y, \dot{x}, \dot{y} \right]$$

are changed into

$$\left[\phi, I, N, \dot{N} \right]$$

The linearization of $\phi(\cdot)$ is not important: it perpetually rotates

The linearization of $I(\cdot)$ is straightforward: $\frac{d}{dt} [\delta I] \equiv 0$

The linearization of $[N, \dot{N}]$ is the famous Hill's equation

$$\frac{d^2}{dt^2} [\delta N] + \Phi(t) \delta N = 0$$

Example: elements of theory of G.W. Hill

In a vicinity of the motion the original coordinates

$$\left[x, y, \dot{x}, \dot{y} \right]$$

are changed into

$$\left[\phi, I, N, \dot{N} \right]$$

The linearization of $\phi(\cdot)$ is not important: it perpetually rotates

The linearization of $I(\cdot)$ is straightforward: $\frac{d}{dt} [\delta I] \equiv 0$

The linearization of $[N, \dot{N}]$ is the famous Hill's equation

$$\frac{d^2}{dt^2} [\delta N] + \Phi(t) \delta N = 0$$

Example: elements of theory of G.W. Hill

In a vicinity of the motion the original coordinates

$$\left[x, y, \dot{x}, \dot{y} \right]$$

are changed into

$$\left[\phi, I, N, \dot{N} \right]$$

The linearization of $\phi(\cdot)$ is not important: it perpetually rotates

The linearization of $I(\cdot)$ is straightforward: $\frac{d}{dt} [\delta I] \equiv 0$

The linearization of $[N, \dot{N}]$ is the famous Hill's equation

$$\frac{d^2}{dt^2} [\delta N] + \Phi(t) \delta N = 0$$

Outcomes of the example

Analysis of dynamics in a vicinity of the motion's orbit requires:

- Decomposition of coordinates into
 - **transverse** to the trajectory ($\dim = 2n - 1$)
 - **along** the trajectory ($\dim = 1$)

In the example they are

$$\left[I, N, \dot{N} \right] \quad \text{and} \quad \phi$$

Outcomes of the example

Analysis of dynamics in a vicinity of the motion's orbit requires:

- Decomposition of coordinates into
 - **transverse** to the trajectory ($\dim = 2n - 1$)
 - **along** the trajectory ($\dim = 1$)

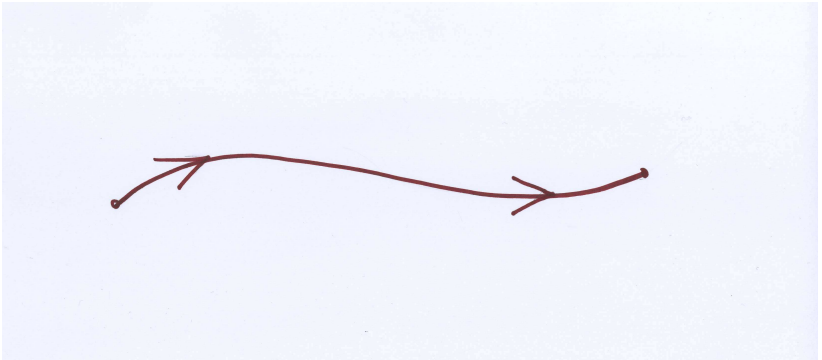
In the example they are

$$\left[I, N, \dot{N} \right] \quad \text{and} \quad \phi$$

-
- Presence of invariants allows to reduce a number of transverse coordinates with non-trivial dynamics.

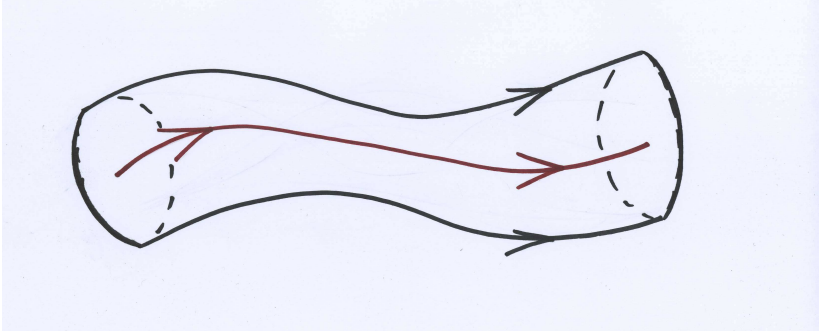
In the example the integral Jacobi $I(\cdot)$ is excluded.

Outcomes of the example: geometrical interpretation



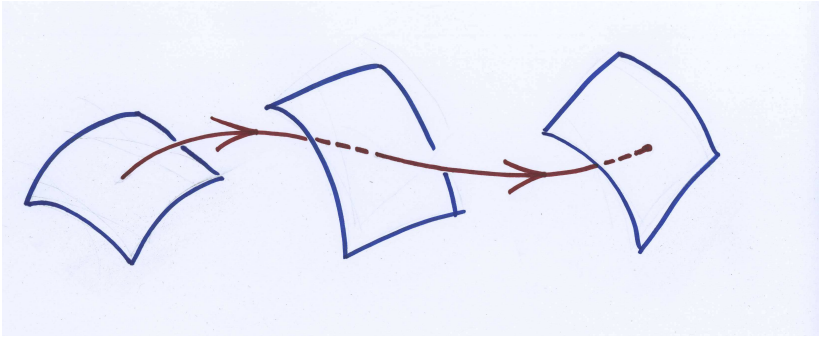
Given a trajectory of a nominal motion

Outcomes of the example: geometrical interpretation



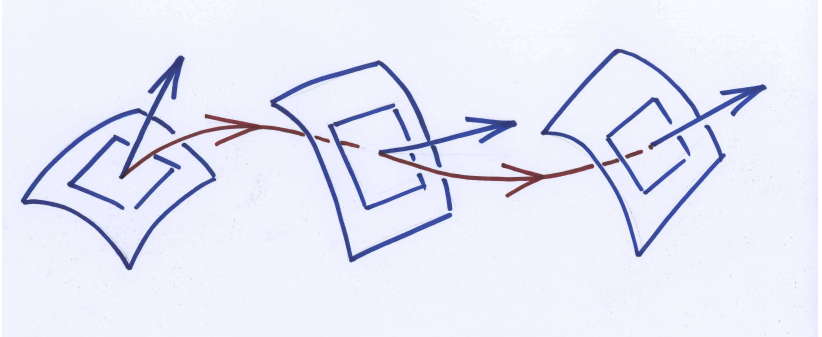
We would like to analyze properties of the dynamics
in its tubing vicinity

Outcomes of the example: geometrical interpretation



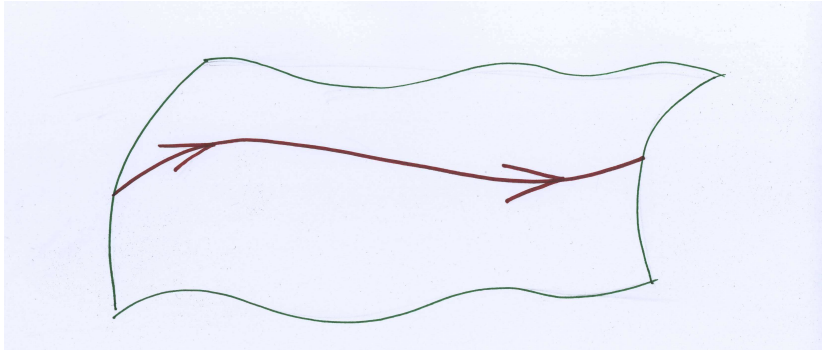
Introduce a family of dis-joint transverse surfaces
that are continuously slicing this vicinity

Outcomes of the example: geometrical interpretation



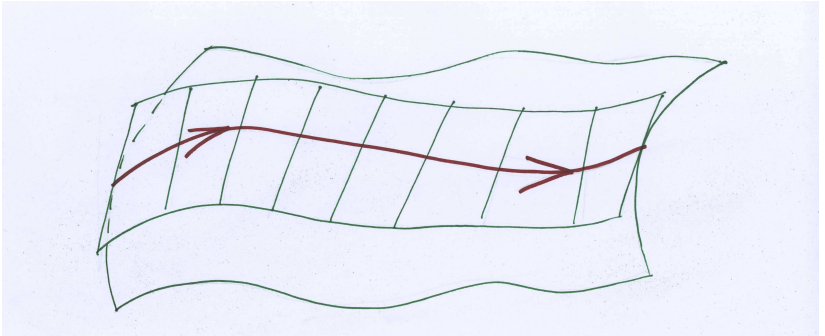
For the linearization of the dynamics the surfaces
are substituted by tangent planes

Outcomes of the example: geometrical interpretation



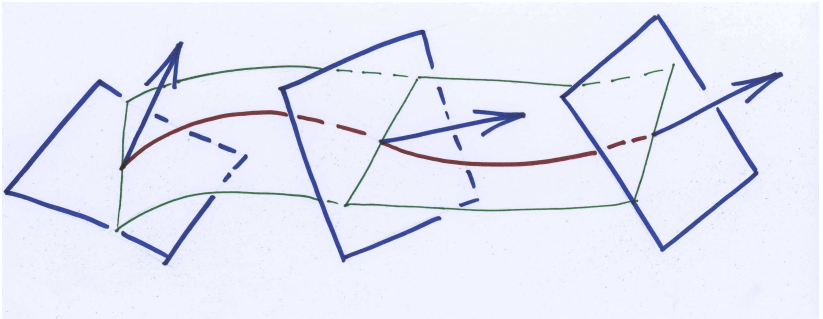
If the dynamics have some invariants,
then they define a manifold

Outcomes of the example: geometrical interpretation



For the linearization we consider the linear subspaces that are tangent to the trajectory along this manifold

Outcomes of the example: geometrical interpretation



Evolution of coordinates on these linear subspaces will define linearization of transverse coordinates with nontrivial behavior