# Lecture 3

## Cases Studies in Motion Planning for Underactuated Systems

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Stockholm, 2018-09-10

**Learning outcomes:** Problem formulation and settings for searching and planning behaviors of mechanical systems with one or two passive degrees of freedom. Examples

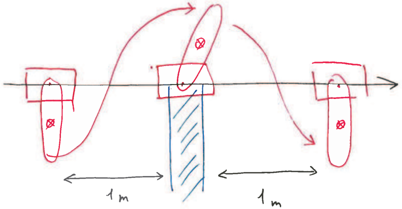
# Outline



# Example: a pendulum on a cart

# Cart-pendulum system: moving a pendulum over a wall

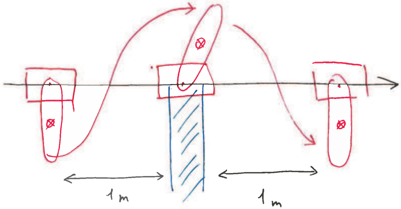



The task is to find a force applied to the cart such that in response the pendulum of the system comes over a wall without collision

One of conceptual solutions:

1. Push the cart to left to swing up the pendulum
2. Push the cart to right to pass the wall keeping the pendulum above the horizontal
3. Bring the cart to the final position simultaneously damping the pendulum oscillations

# Cart-pendulum system: moving a pendulum over a wall

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# Cart-pendulum system: moving a pendulum over a wall

For the step 1 the dynamics with constant force $f_1 < 0$ are

$$2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 = f_1$$

$$\cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta = 0$$

$f_1$ should be sufficient to swing the pendulum above the horizontal

To find $f_1$, one needs to decouple the dynamics of $x$ and $\theta$

E.g. one can take $\ddot{x}$ from the 1st eqn and plug it into the 2nd eqn

$$\cos\theta \cdot \underbrace{\frac{1}{2}\left(f_1 - \cos\theta \cdot \ddot{\theta} + \sin\theta \cdot \dot{\theta}^2\right)}_{=\ddot{x}} + \ddot{\theta} - g \cdot \sin\theta = 0$$

Collecting the terms results in the dynamics of $\theta$-variable

$$\left(1 - \frac{1}{2}\cos^2\theta\right) \cdot \ddot{\theta} + \frac{1}{2}\cos\theta \cdot \sin\theta \cdot \dot{\theta}^2 + \left(\frac{1}{2} \cdot f_1 \cdot \cos\theta - g \cdot \sin\theta\right) = 0$$

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# Cart-pendulum system: moving a pendulum over a wall

The initial conditions for a searched motion $\theta^*(\cdot)$ of the system

$$\left(1 - \tfrac{1}{2}\cos^2\theta\right)\cdot\ddot{\theta} + \tfrac{1}{2}\cos\theta\cdot\sin\theta\cdot\dot{\theta}^2 + \left(\tfrac{1}{2}\cdot f_1\cdot\cos\theta - g\cdot\sin\theta\right) = 0$$

are defined as

$$\theta^*(0) = \pi, \quad \dot{\theta}^*(0) = 0$$

---

How to find $f_1$ such that the solution $\theta^*(\cdot)$ will pass through

$$\theta^*(T) = a, \quad \dot{\theta}^*(T) = 0 \text{ ???}$$

---

The $\theta$-dynamics has the integral of motion for any $f_1$ defined as

$$E_{red}(\theta(t), \dot{\theta}(t)) = \tfrac{1}{2}\left(1 - \tfrac{1}{2}\cos^2\theta(t)\right)\dot{\theta}^2(t) + g\cos\theta(t) + \tfrac{1}{2}f_1\sin\theta(t) \equiv C$$

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Hence

$$\begin{aligned} E_{red}(\theta^*(T),\dot\theta^*(T)) &= E_{red}(a,0) = g\cdot\cos a + \tfrac{1}{2}\cdot f_1\cdot\sin a \\ &= E_{red}(\theta^*(0),\dot\theta^*(0)) = g\cdot\cos\pi + \tfrac{1}{2}\cdot f_1\cdot\sin\pi \end{aligned}$$

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Hence

$$g\cdot\cos a + \frac{1}{2}\cdot f_1\cdot\sin a = -g$$

$$f_1 = -2\cdot\frac{1+\cos a}{\sin a}\cdot g$$

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**How to compute a time constant T?**

# Cart-pendulum system: moving a pendulum over a wall

To estimate the position and velocity of the cart at the end of this maneuver, one can observe that

$$
\begin{aligned}
2\cdot\ddot{x} + \cos\theta\cdot\ddot{\theta} - \sin\theta\cdot\dot{\theta}^2 &= f_1 \\
&\Updownarrow \\
\frac{d}{dt}\left[2\cdot\dot{x} + \cos\theta\cdot\dot{\theta}\right] &= f_1 \\
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\left\{2\cdot\dot{x}(t) + \cos\theta(t)\cdot\dot{\theta}(t)\right\} - \left\{2\cdot\dot{x}(0) + \cos\theta(0)\cdot\dot{\theta}(0)\right\} &= f_1\cdot t \\
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\left\{2\cdot\dot{x}(0) + \cos\theta(0)\cdot\dot{\theta}(0)\right\}\cdot t &= \frac{1}{2}\cdot f_1\cdot t^2
\end{aligned}
$$

If the system was at rest in the beginning of the movement, then

$$
2x(T) + \sin a - 2x(0) = \frac{1}{2}\cdot f_1\cdot T^2, \qquad 2\dot{x}(T) = f_1\cdot T
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# Cart-pendulum system: moving a pendulum over a wall

On the 2nd step, one needs to find a constant $f_2 > 0$ such that for the solution of the system

$$\begin{aligned} 2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 &= f_2 \\ \cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta &= 0 \end{aligned}$$

the position of the cart moves behind the wall while the pendulum stays above the horizontal for the same time interval.

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As before the behavior of the $\theta$-variable is defined by as a solution $\theta^*(\cdot)$ of the reduced dynamics

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with initial conditions at

$$\theta^*(0) = a, \quad \dot{\theta}^*(0) = 0 \quad \text{with} \quad -\frac{\pi}{2} < a < \frac{\pi}{2}$$

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As for the 1st maneuver, one can use the integral of $\theta$-dynamics

$$E_{red}(\theta(t), \dot{\theta}(t)) = \frac{1}{2}\left(1 - \frac{1}{2}\cos^2\theta(t)\right)\dot{\theta}^2(t) + g\cos\theta(t) + \frac{1}{2} f_2 \sin\theta(t) \equiv C$$

for computing for the second maneuver

- time to reach the critical values of the angle: $\theta_c = \pm\frac{\pi}{2}$
- angular velocity of the pendulum at critical values of $\theta$

as functions of the input variable $f_2$.

---

**Homework:** find constants $f_1$ and $f_2$, which in combination create the requested behavior of the cart-pendulum.

---

One can investigate more complicated situations when

$$f_1 = f_{10} + f_{11}\cdot\theta + \cdots + f_{1k}\cdot\theta^k, \quad f_2 = f_{20} + f_{21}\cdot\theta + \cdots + f_{2m}\cdot\theta^m$$

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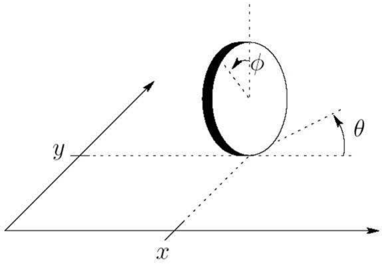
# A coin rolling on a table

# Example: planning a rolling of a coin



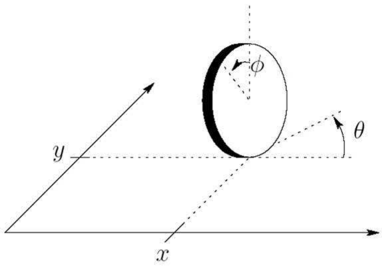

The equations of motion are

$$m\ddot{x} = F_x^c, \qquad m\ddot{y} = F_y^c, \qquad J\ddot{\theta} = u$$

Here $u$ is control; $F^c = [F_x^c;\ F_y^c]$ is the force due to the constraint

$$\dot{y}(t)\cos\theta(t) - \dot{x}(t)\sin\theta(t) \equiv 0, \quad \forall\, t$$

# Example: planning a rolling of a coin

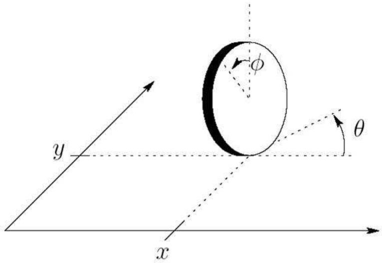



The equations of motion are

$$m\ddot{x} = \lambda \cdot \cos(\theta - \tfrac{\pi}{2}), \qquad m\ddot{y} = \lambda \cdot \sin(\theta - \tfrac{\pi}{2}), \qquad J\ddot{\theta} = u$$

Here $\lambda$ is amplitude of the constraint force.

# Example: planning a rolling of a coin

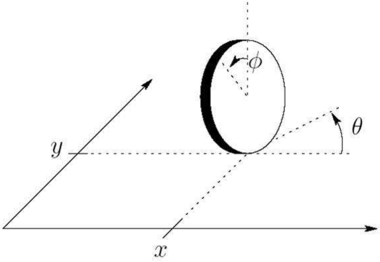



The equations of motion are

$$\begin{aligned}
\ddot{x} &= -\left[\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta\right]\cdot\dot{\theta}\cdot\sin(\theta)\\
\ddot{y} &= \left[\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta\right]\cdot\dot{\theta}\cdot\cos(\theta)\\
J\ddot{\theta} &= u
\end{aligned}$$

where constraint force is quadratic in velocities.

# Example: planning a rolling of a coin




We have already seen irreducible equations of motion of the form

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}\mathcal{L}\right] - \frac{\partial}{\partial q}\mathcal{L} = R(q,\dot{q}) + B(q)u, \qquad R_i = \dot{q}^T r_i(q)\dot{q}$$

Here $q \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $R(\cdot)$ is a vector of reaction forces.

# Example: planning a rolling of a coin of along a circle

Motion planning for the dynamical model

$$\begin{aligned} \ddot{x} &= -[\dot{y} \cdot \sin\theta + \dot{x} \cdot \cos\theta] \cdot \dot{\theta} \cdot \sin(\theta) \\ \ddot{y} &= [\dot{y} \cdot \sin\theta + \dot{x} \cdot \cos\theta] \cdot \dot{\theta} \cdot \cos(\theta) \\ J\ddot{\theta} &= u \end{aligned}$$

can be quite non-trivial.

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Find feasible motions of the system consistent with requirement

the center of mass should stay on a circle of radius $R$

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can be quite non-trivial.

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I.e. along any such motion $[x_c(t), y_c(t), \theta_c(t)]$ the relations hold

$$x_c(t) = R\cdot\cos\left(\theta_c(t) - \frac{\pi}{2}\right) \qquad y_c(t) = R\cdot\sin\left(\theta_c(t) - \frac{\pi}{2}\right)$$

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I.e. along any such motion $[x_c(t), y_c(t), \theta_c(t)]$ the relations hold

$$x_c(t) = R\cdot\sin\theta_c(t) \qquad y_c(t) = -R\cdot\cos\theta_c(t)$$

## Example: planning a rolling of a coin of along a circle

Motion planning for the dynamical model

$$\begin{aligned} \ddot{x} &= -[\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta]\cdot\dot{\theta}\cdot\sin(\theta) \\ \ddot{y} &= [\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta]\cdot\dot{\theta}\cdot\cos(\theta) \\ J\ddot{\theta} &= u \end{aligned}$$

can be quite non-trivial.

---

I.e. along any such motion $[x_c(t), y_c(t), \theta_c(t)]$ the relations hold

$$x_c(t) = R\cdot\sin\theta_c(t) \qquad y_c(t) = -R\cdot\cos\theta_c(t)$$

$$\dot{x}_c(t) = R\cdot\cos\theta_c(t)\cdot\dot{\theta}_c(t) \qquad \dot{y}_c(t) = R\cdot\sin\theta_c(t)\cdot\dot{\theta}_c(t)$$

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$$\ddot{x}_c = R\left[\cos\theta_c\ddot{\theta}_c - \sin\theta_c\dot{\theta}_c^2\right] \qquad \ddot{y}_c = R\left[\sin\theta_c\ddot{\theta}_c + \cos\theta_c\dot{\theta}_c^2\right]$$

# Example: planning a rolling of a coin of along a circle

Along a circular motion $[x_c(t), y_c(t), \theta_c(t)]$ of the system

$$\begin{aligned} \ddot{x} &= -[\dot{y} \cdot \sin\theta + \dot{x} \cdot \cos\theta] \cdot \dot{\theta} \cdot \sin(\theta) \\ \ddot{y} &= [\dot{y} \cdot \sin\theta + \dot{x} \cdot \cos\theta] \cdot \dot{\theta} \cdot \cos(\theta) \\ J\ddot{\theta} &= u \end{aligned}$$

the relations hold

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the relations hold

$$\begin{aligned} \cos\theta_c \cdot \ddot{x}_c &= \cos\theta_c \cdot R \cdot \left[\cos\theta_c \ddot{\theta}_c - \sin\theta_c \dot{\theta}_c^2\right] \\ \sin\theta_c \cdot \ddot{y}_c &= \sin\theta_c \cdot R \cdot \left[\sin\theta_c \ddot{\theta}_c + \cos\theta_c \dot{\theta}_c^2\right] \end{aligned}$$

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$$\begin{aligned}\cos\theta_c\cdot\ddot{x}_c &= \cos\theta_c\cdot R\cdot\left[\cos\theta_c\ddot{\theta}_c - \sin\theta_c\dot{\theta}_c^2\right]\\ \sin\theta_c\cdot\ddot{y}_c &= \sin\theta_c\cdot R\cdot\left[\sin\theta_c\ddot{\theta}_c + \cos\theta_c\dot{\theta}_c^2\right]\end{aligned}$$

$$\Downarrow$$

$$\cos\theta_c\cdot\ddot{x}_c + \sin\theta_c\cdot\ddot{y}_c = R\ddot{\theta}_c$$

# Example: planning a rolling of a coin of along a circle

Along a circular motion $[x_c(t), y_c(t), \theta_c(t)]$ of the system

$$
\begin{aligned}
\ddot{x} &= -[\dot{y} \cdot \sin\theta + \dot{x} \cdot \cos\theta] \cdot \dot{\theta} \cdot \sin(\theta) \\
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J\ddot{\theta} &= u
\end{aligned}
$$

the relations hold

$$
\begin{aligned}
\cos\theta_c \cdot \ddot{x}_c &= \cos\theta_c \cdot R \cdot \left[\cos\theta_c \ddot{\theta}_c - \sin\theta_c \dot{\theta}_c^2\right] \\
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& \Downarrow \\
\cos\theta_c \cdot \ddot{x}_c + \sin\theta_c \cdot \ddot{y}_c &= R \cdot \ddot{\theta}_c \\
& \Downarrow \\
0 &= R \cdot \ddot{\theta}_c
\end{aligned}
$$

# Example: planning a rolling of a coin of along a circle

Any circular motion $[x_c(t), y_c(t), \theta_c(t)]$ of the system

$$\begin{aligned} \ddot{x} &= -[\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta]\cdot\dot{\theta}\cdot\sin(\theta) \\ \ddot{y} &= [\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta]\cdot\dot{\theta}\cdot\cos(\theta) \\ J\ddot{\theta} &= u \end{aligned}$$

has the form

$$\begin{aligned} \theta_c(t) &= \omega_c\cdot t + \theta_0 \\ x_c(t) &= R\cdot\sin\theta_c(t) \\ y_c(t) &= -R\cdot\cos\theta_c(t) \\ u_c(t) &= 0 \end{aligned}$$

How to find new coordinates for the dynamics that are zero on the nominal motion and nonzero away from it? Regulating them to zero helps stabilizing such motion orbitally

## Example: planning a rolling of a coin of along a circle

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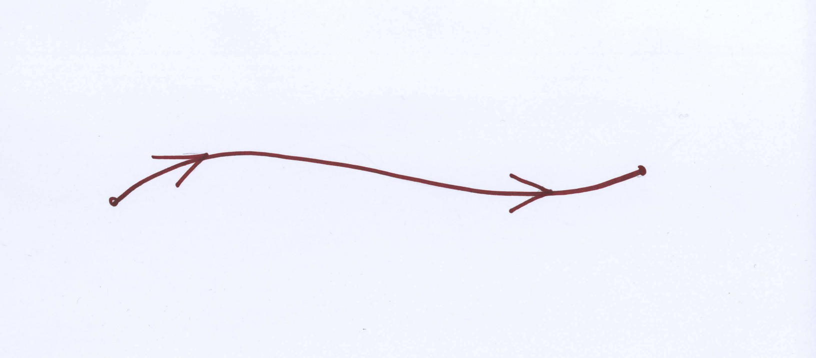
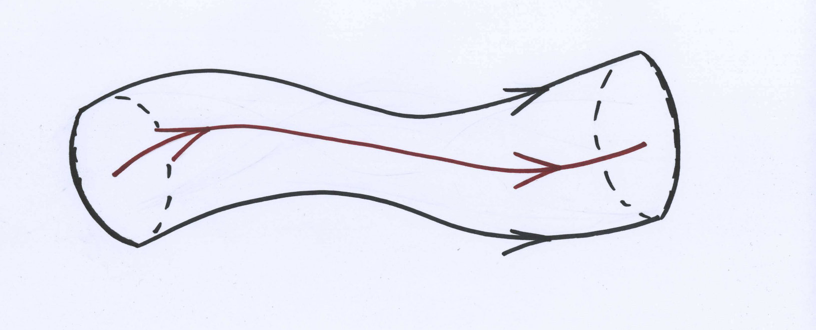
---

**How to find new coordinates for the dynamics that are zero on the nominal motion and nonzero away from it? Regulating them to zero helps stabilizing such motion orbitally**



Given a trajectory of a nominal motion

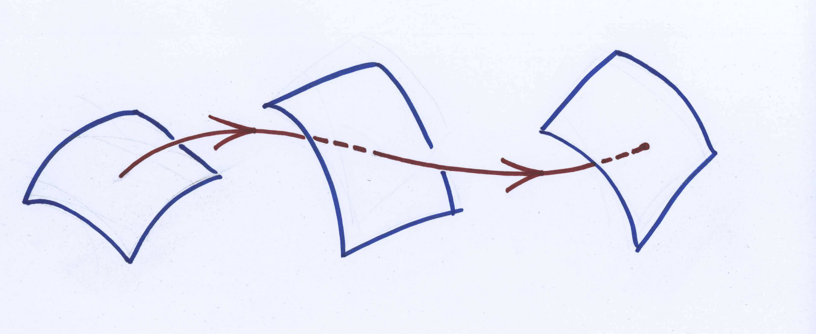
## Geometrical interpretation



We would like to analyze properties of the dynamics

in its tubing vicinity

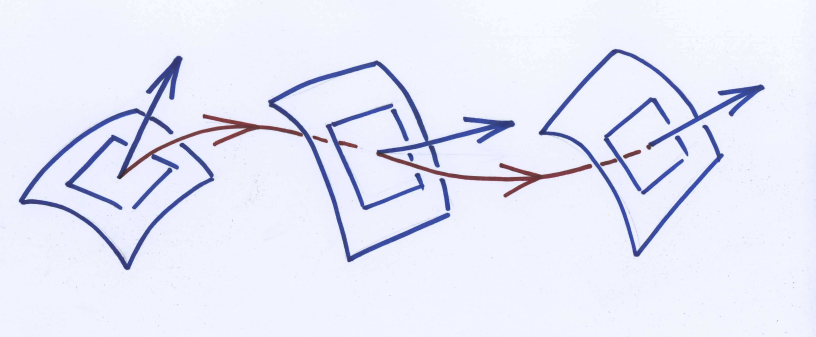
# Geometrical interpretation



Introduce a family of dis-joint transverse surfaces

that are continuously slicing this vicinity

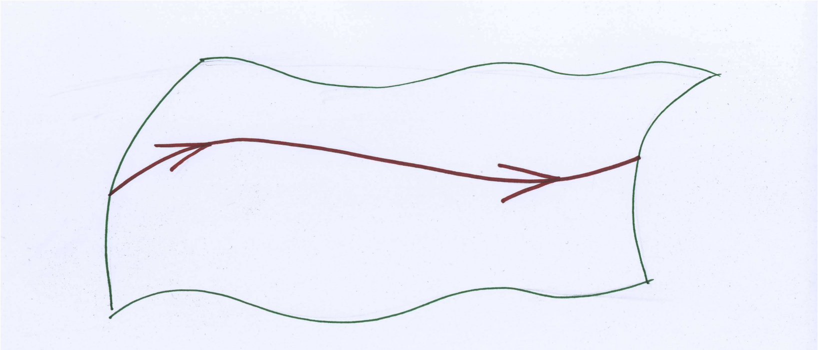
# Geometrical interpretation



For the linearization of the dynamics the surfaces

are substituted by tangent planes

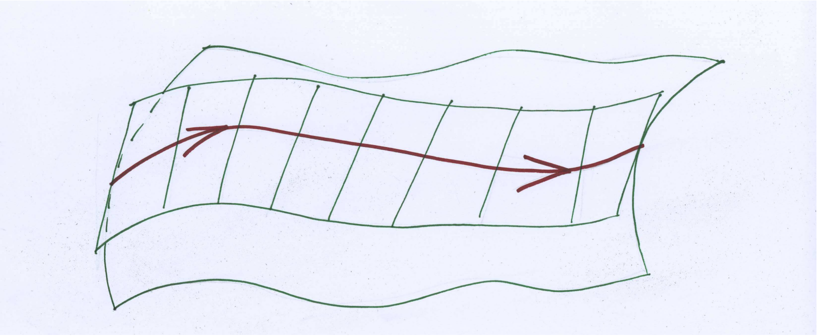
# Geometrical interpretation



If the dynamics have some invariants,

then they define a manifold

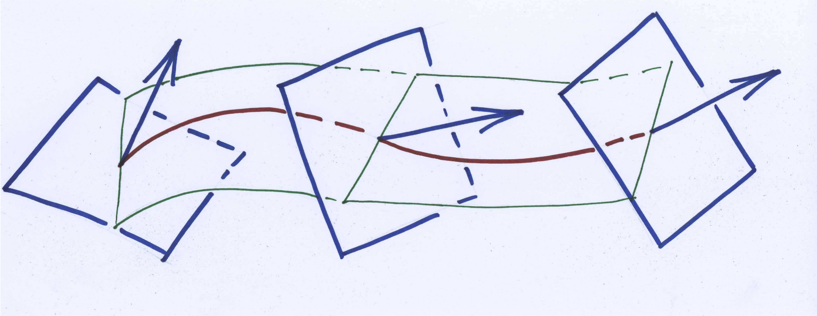
# Geometrical interpretation



For the linearization we consider the linear subspaces

that are tangent to to the trajectory along this manifold

# Geometrical interpretation

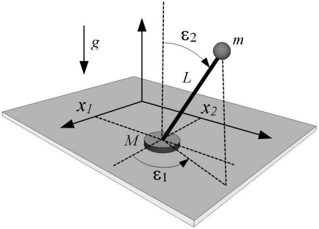


Evolution of coordinates on these linear subspaces will define

linearization of transverse coordinates with nontrivial behavior

# Example: a spherical pendulum on a puck

# Example: motions of a spherical pendulum on a puck




The system has four generalized coordinates: $(x_1, x_2)$ are variables for representing the position of the puck on a plane; $(\varepsilon_1, \varepsilon_2)$ are two angles (precession and nutation) for representing the status of the pendulum.

The dynamics of the system are

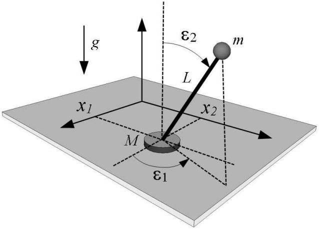
$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\varepsilon}_1}\right] - \frac{\partial \mathcal{L}}{\partial \varepsilon_1} = 0 \qquad \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\varepsilon}_2}\right] - \frac{\partial \mathcal{L}}{\partial \varepsilon_2} = 0$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right] - \frac{\partial \mathcal{L}}{\partial x_1} = F_1 \qquad \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right] - \frac{\partial \mathcal{L}}{\partial x_2} = F_2$$

with $F_1$, $F_2$ being external forces acting on the puck.

**The task: to find behaviors of the system when the puck is forced to move along of a circle of a radius $R$**

# Example: motions of a spherical pendulum on a puck

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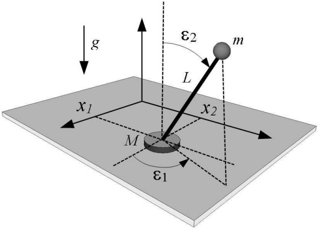


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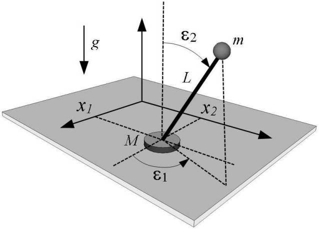



The task means that for any searched behavior in polar coordinates on a plane

$$x_1(t) = R\cdot\cos\psi(t)$$
$$x_2(t) = R\cdot\sin\psi(t)$$

while the behaviors of $\varepsilon_1(\cdot)$ and $\varepsilon_2(\cdot)$ are

$$\varepsilon_1(t) = \phi_1\left(\psi(t)\right), \quad \varepsilon_2(t) = \phi_2\left(\psi(t)\right)$$

with some functions $\phi_1(\cdot)$, $\phi_2(\cdot)$.

Suppose that the angle $\varepsilon_1$ can be chosen as a motion generator and on a particular behavior

$$q^*(t) = [x_1^*(t), x_2^*(t), \varepsilon_1^*(t), \varepsilon_2^*(t)]$$

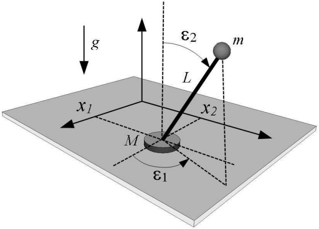
the relation between $\psi(\cdot)$ and $\varepsilon_1^*(\cdot)$ is

$$\psi(t) = \varepsilon_1^*(t).$$

**What are then the equations of motion for $q^*(\cdot)$?**

# Example: motions of a spherical pendulum on a puck

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The assumption means that

$$\begin{aligned} x_1^*(t) &= R \cdot \cos \varepsilon_1^*(t) \\ x_2^*(t) &= R \cdot \sin \varepsilon_1^*(t) \end{aligned}$$

---

Furthermore it means that the equations of motion

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot\varepsilon_1}\right] - \frac{\partial \mathcal{L}}{\partial \varepsilon_1} &= 0 \qquad & \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot\varepsilon_2}\right] - \frac{\partial \mathcal{L}}{\partial \varepsilon_2} &= 0 \\ \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot x_1}\right] - \frac{\partial \mathcal{L}}{\partial x_1} &= F_1 \qquad & \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot x_2}\right] - \frac{\partial \mathcal{L}}{\partial x_2} &= F_2 \end{aligned}$$

for this behavior become

$$\begin{aligned} \Big(R + L \cdot \sin(\varepsilon_2)\Big)\,\ddot\varepsilon_1 + 2 \cdot L \cdot \cos(\varepsilon_2) \cdot \dot\varepsilon_1 \cdot \dot\varepsilon_2 &= 0 \\ L \cdot \ddot\varepsilon_2 - \cos(\varepsilon_2) \cdot \Big(R + L \cdot \sin(\varepsilon_2)\Big)\,\dot\varepsilon_1^2 - g \cdot \sin(\varepsilon_2) &= 0 \end{aligned}$$

---

**How to compute $\varepsilon_1^*(\cdot)$ and $\varepsilon_2^*(\cdot)$?**

## Example: motions of a spherical pendulum on a puck

The assumption means that

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\begin{aligned}
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# Example: motions of a spherical pendulum on a puck

The first equation of the system

$$\begin{aligned} \Big(R + L\cdot\sin(\varepsilon_2)\Big)\,\ddot\varepsilon_1 + 2\cdot L\cdot\cos(\varepsilon_2)\cdot\dot\varepsilon_1\cdot\dot\varepsilon_2 &= 0 \\ L\cdot\ddot\varepsilon_2 - \cos(\varepsilon_2)\cdot\Big(R + L\cdot\sin(\varepsilon_2)\Big)\,\dot\varepsilon_1^2 - g\cdot\sin(\varepsilon_2) &= 0 \end{aligned}$$

can be written as

$$\frac{d}{dt}\Big[\left(R + L\cdot\sin(\varepsilon_2)\right)^2\dot\varepsilon_1\Big] = 0$$

---

$$\left[R + L\cdot\sin(\varepsilon_2(t))\right]^2\dot\varepsilon_1(t) = \left[R + L\cdot\sin(\varepsilon_2(0))\right]^2\dot\varepsilon_1(0) = C$$

---

Therefore, the second equation can be rewritten as

$$L\cdot\ddot\varepsilon_2 - \cos(\varepsilon_2)\cdot\frac{C}{\Big(R + L\cdot\sin(\varepsilon_2)\Big)^3} - g\cdot\sin(\varepsilon_2) = 0$$

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# Example: motions of a spherical pendulum on a puck

How to find solutions of the system

$$L \cdot \ddot{\varepsilon}_2 - \cos(\varepsilon_2) \cdot \frac{C}{\Big(R + L \cdot \sin(\varepsilon_2)\Big)^3} - g \cdot \sin(\varepsilon_2) = 0 \ ???$$

---

How to find the solutions that stay above the horizontal, i.e.

$$-\frac{\pi}{2} < \varepsilon_2(t) < \frac{\pi}{2} \quad \forall t \quad ???$$

---

The system has a conserved quantity

$$E_2 = \tfrac{1}{2}\,\dot{\varepsilon}_2^2 + \frac{g}{L}\cos(\varepsilon_2) + \frac{C^2}{2L^2\left[R + L\sin(\varepsilon_2)\right]^2}$$

Analysis of level sets of $E_2(\cdot)$ allows answering both questions

# Example: motions of a spherical pendulum on a puck

How to find solutions of the system

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# Example: motions of a spherical pendulum on a puck

How to find solutions of the system

$$L \cdot \ddot{\varepsilon}_2 - \cos(\varepsilon_2) \cdot \frac{C}{\Big(R + L \cdot \sin(\varepsilon_2)\Big)^3} - g \cdot \sin(\varepsilon_2) = 0 \text{ ???}$$

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How to find the solutions that stay above the horizontal, i.e.

$$-\frac{\pi}{2} < \varepsilon_2(t) < \frac{\pi}{2} \quad \forall t \quad \text{???}$$

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The system has a conserved quantity

$$E_2 = \tfrac{1}{2}\,\dot{\varepsilon}_2^2 + \frac{g}{L}\cos(\varepsilon_2) + \frac{C^2}{2L^2\left[R + L\sin(\varepsilon_2)\right]^2}$$

Analysis of level sets of $E_2(\cdot)$ allows answering both questions