

# Lecture 4

## Motion Representations for Underactuated Systems

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Anton Shiriaev

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**Learning outcomes:** Concepts of a motion generator (MG) and its dynamics for mechanical systems. A nested representation of motion candidates for underactuated mechanical systems. Properties of the dynamics of a MG derived based on the nested representation of a feasible behavior of an underactuated mechanical system. Examples

1. Example: a pendulum on a cart system
  - 1.1 Nested representation of a motion
2. Dynamics of the motion generator
  - 2.1 Integrability
  - 2.2 Conditions for presence of a cycle
  - 2.3 Integral is a distance
  - 2.4 Passivity relation for reduced dynamics
3. Example: a cart-pendulum system
  - 3.1 Moving a cart-pendulum over an obstacle

**Example: a pendulum on a cart system**

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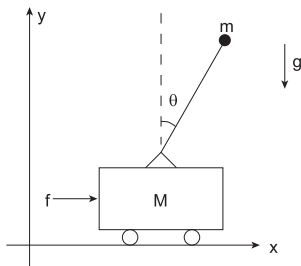
# Nested representation of motions for cart-pendulum system

Given a forced motion of the system

$$[x^*(t), \theta^*(t), f^*(t)], t \in [0, T],$$

assume that it admits alternative representation

$$x^*(t) = \phi(\theta^*(t)), t \in [0, T]$$



The coupled dynamics of the a pendulum on a cart are

$$2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 = f$$

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

What would be then a differential equation  
for describing the dynamics of  $\theta$ -variable?

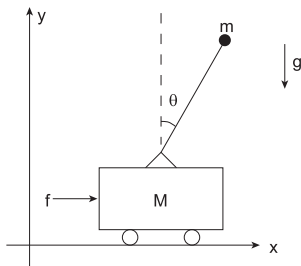
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# Nested representation of motions for cart-pendulum system

The relation

$$x^*(t) = \phi(\theta^*(t)), \quad t \in [0, T]$$

implies the corresponding relations on the 1st and 2nd derivatives

$$\dot{x}^*(t) = \phi'(\theta^*(t))\dot{\theta}^*(t), \quad \ddot{x}^*(t) = \phi''(\theta^*(t))\dot{\theta}^*(t)\dot{\theta}^*(t) + \phi'(\theta^*(t))\ddot{\theta}^*(t)$$

Substituting the relations into the non-actuated part of the dynamics

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

results in the decoupled dynamics for  $\theta$

$$\cos \theta \cdot \left( \phi''(\theta)\dot{\theta}^2 + \phi'(\theta)\ddot{\theta} \right) + \ddot{\theta} - g \cdot \sin \theta = 0$$

which is more transparent after collecting the similar terms

$$\left[ 1 + \cos \theta \cdot \phi'(\theta) \right] \ddot{\theta} + \cos \theta \cdot \phi''(\theta) \cdot \dot{\theta}^2 - g \cdot \sin \theta = 0$$

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## Nested representation of motions for cart-pendulum system

The motion generator for the behavior can be chosen differently

$$\theta^*(t) = \phi(x^*(t)), \quad t \in [0, T]$$

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**What are properties of that  $(\alpha\text{-}\beta\text{-}\gamma)$  system?**

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## **Dynamics of the motion generator**

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## Integrability of dynamics of the motion generator

Suppose the solution

$$\theta(t) = \theta(t, \theta_0, \dot{\theta}_0)$$

of the system

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

with initial conditions  $[\theta_0, \dot{\theta}_0]$  exists.

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Then the function

$$I(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0) = \dot{\theta}^2 - \psi(\theta_0, \theta) \left[ \dot{\theta}_0^2 - \int_{\theta_0}^{\theta} \psi(s, \theta_0) \frac{2\gamma(s)}{\alpha(s)} ds \right]$$

with

$$\psi(\theta_0, \theta_1) = \exp \left\{ -2 \int_{\theta_0}^{\theta_1} \frac{\beta(\tau)}{\alpha(\tau)} d\tau \right\}$$

preserves its **zero-value** along this (even unbounded) solution

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## Example

Consider the system

$$\ddot{\theta} + \sin \theta = 0$$

that is

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

with

$$\alpha(\theta) = 1, \quad \beta(\theta) = 0, \quad \gamma(\theta) = \sin \theta$$

Then

$$\begin{aligned} I &= \dot{\theta}^2 - \exp \left\{ -2 \int_{\theta_0}^{\theta} \frac{\beta(\tau)}{\alpha(\tau)} d\tau \right\} \left[ \dot{\theta}_0^2 - \int_{\theta_0}^{\theta} \exp \left\{ 2 \int_{\theta_0}^s \frac{\beta(\tau)}{\alpha(\tau)} d\tau \right\} \frac{2\gamma(s)}{\alpha(s)} ds \right] \\ &= \dot{\theta}^2 - \dot{\theta}_0^2 + \int_{\theta_0}^{\theta} 2 \sin s ds \\ &= \left( \dot{\theta}^2 - 2 \cos \theta \right) - \left( \dot{\theta}_0^2 - 2 \cos \theta_0 \right) \end{aligned}$$

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## Extension of Lyapunov lemma

Let  $\theta_0$  be an equilibrium of the system

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that is, the solution of the equation  $\gamma(\theta_0) = 0$ .

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Suppose  $\alpha(\cdot)$  and  $\gamma(\cdot)$  are  $C^1$  at  $\theta_0$ , and  $\beta(\cdot)$  is continuously differentiable in some vicinity of  $\theta_0$ , but not necessarily at  $\theta_0$ .

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Consider the second order auxiliary linear system (the linearization of nonlinear one at  $\theta_0$ , if  $\beta(\cdot)$  would be  $C^1$ -smooth at  $\theta_0$ )

$$\frac{d^2}{dt^2}z + \left[ \frac{d}{d\theta} \frac{\gamma(\theta)}{\alpha(\theta)} \right] \Big|_{\theta=\theta_0} \cdot z = 0$$

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If the auxiliary linear system has the center at  $z = 0$ , then the nonlinear system has the center at the equilibrium  $\theta_0$

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## Example

In the first Lecture, we have seen the system

$$\ddot{\theta} + \left[ \sin(100|\theta|) - \sqrt{|\theta|} \right] \dot{\theta}^2 + a \cdot \sin \theta = 0$$

Its linearization at the equilibrium

$$\theta_0 = 0$$

does not exist and one **cannot** apply Lyapunov lemma for this system: **its coefficients are not analytic functions.**

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But the statement says that, in fact, this system has the centre at

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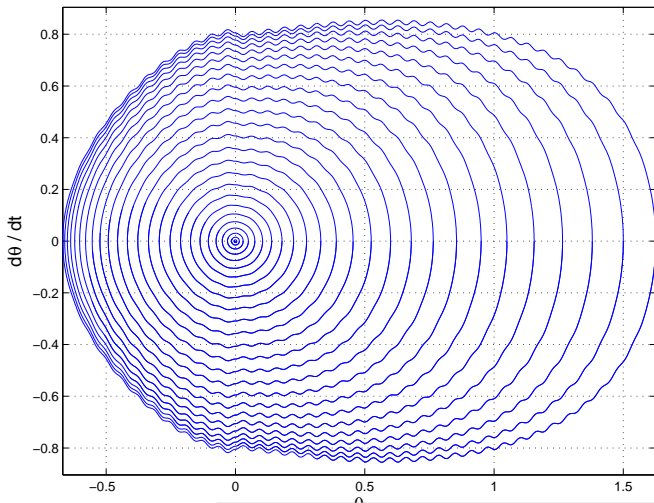
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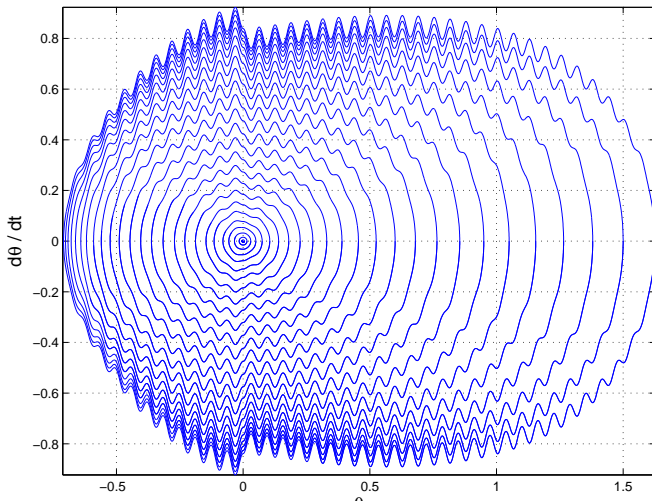
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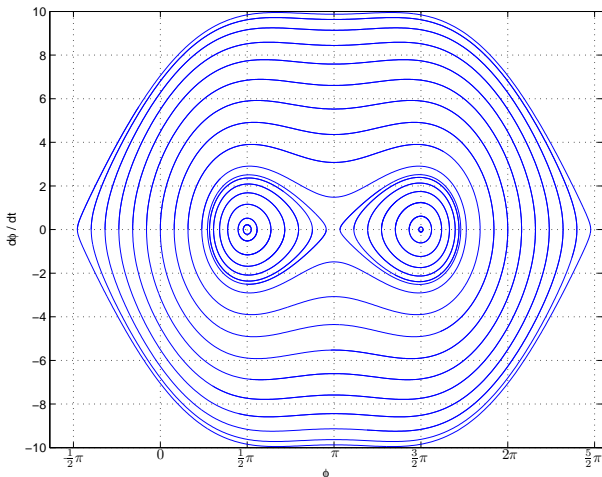
The phase portrait of  $\ddot{\theta} + [\sin(100|\theta|) - \sqrt{|\theta|}] \dot{\theta}^2 + \sin \theta = 0$

## Example



The phase portrait of  $\ddot{\theta} + [5 \cdot \sin(100|\theta|) - \sqrt{|\theta|}] \dot{\theta}^2 + \sin \theta = 0$

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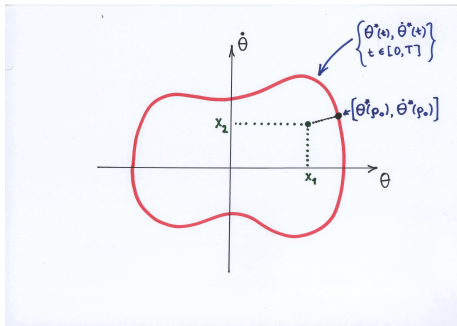
The phase portrait of a reduced system with large cycles surrounding several equilibria.

## Integral is a distance

Given the target orbit  $[\theta^*(t), \dot{\theta}^*(t)]$ , then

- For any  $x_1$  and  $x_2$  the function  $I(\cdot)$  satisfies the identity

$$I(x_1, x_2, \theta^*(0), \dot{\theta}^*(0)) \equiv I(x_1, x_2, \theta^*(\rho), \dot{\theta}^*(\rho)), \quad \rho \in [0, T]$$



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- Nearby the target orbit  $|I(\cdot)|$  measures the distance to the orbit. Namely, the following approximation holds

$$\begin{aligned} I(x_1, x_2, \theta^*(\rho_0), \dot{\theta}^*(\rho_0))^2 &= \\ &= \min_{0 \leq \rho < T} \left\{ |x_1 - \theta^*(\rho)|^2 + |x_2 - \dot{\theta}^*(\rho)|^2 \right\} \times \\ &\quad \times 4 \left[ \dot{\theta}^*(\rho_0)^2 + \ddot{\theta}^*(\rho_0)^2 \right] + \dots \end{aligned}$$

Here

$$\rho_0 = \arg \min_{0 \leq \rho < T} \left\{ |x_1 - \theta^*(\rho)|^2 + |x_2 - \dot{\theta}^*(\rho)|^2 \right\}$$

## Passivity relation for reduced dynamics

The time derivative of the function  $I(\theta, \dot{\theta}, \mathbf{x}, \mathbf{y})$  defined as

$$I = \dot{\theta}^2 - \exp \left\{ -2 \int_{\mathbf{x}}^{\theta} \frac{\beta(\tau)}{\alpha(\tau)} d\tau \right\} \left[ \mathbf{y}^2 - \int_{\mathbf{x}}^{\theta} \exp \left\{ 2 \int_{\mathbf{x}}^s \frac{\beta(\tau)}{\alpha(\tau)} d\tau \right\} \frac{2\gamma(s)}{\alpha(s)} ds \right]$$

with  $\mathbf{x}$  and  $\mathbf{y}$  being some constants, calculated along a solution of the system

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = W$$

has the form

$$\frac{d}{dt} I(\theta, \dot{\theta}, \mathbf{x}, \mathbf{y}) = \dot{\theta} \left\{ \frac{2}{\alpha(\theta)} W - \frac{2\beta(\theta)}{\alpha(\theta)} I(\theta, \dot{\theta}, \mathbf{x}, \mathbf{y}) \right\}$$

## **Example: a cart-pendulum system**

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## Planning oscillations of a cart-pendulum system

A searched oscillation of the pendulum around the upright equilibrium for the system

$$2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 = u$$

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

can be written through the nested representation as

- $\theta^*(\cdot)$  defined as a solution of the reduced dynamics

$$\left[1 + \cos \theta \cdot \phi'(\theta)\right] \ddot{\theta} + \cos \theta \cdot \phi''(\theta) \cdot \dot{\theta}^2 - g \cdot \sin \theta = 0$$

- $x^*(\cdot)$  defined by the kinematic relation

$$x^*(t) = \phi(\theta^*(t))$$

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The system

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has the center at  $\theta_e = 0$ , if the following linear system

$$\ddot{z} + \left[ \frac{d}{d\theta} \left( \frac{-g \cdot \sin \theta}{1 + \cos \theta \cdot \phi'(\theta)} \right) \right] \Big|_{\theta=0} \cdot z = 0$$

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The calculations show that the linearization has the form

$$\ddot{z} + \omega^2 z = 0$$

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$$\frac{-g}{1 + \phi'(0)} = \omega^2 \quad \left( \Rightarrow \phi'(0) = - \left[ 1 + \frac{g}{\omega^2} \right] \right)$$

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## Planning oscillations of a cart-pendulum system

Consider the following function

$$x = \phi(\theta) = - \left[ 1 + \frac{g}{\omega^2} \right] \ln \left( \frac{1 + \sin \theta}{\cos \theta} \right)$$

where  $\omega$  is the desired frequency of oscillations.

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One can check that for this particular choice

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## Planning oscillations of a cart-pendulum system

The dynamics of the cart-pendulum system

$$\begin{aligned}2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 &= u \\ \cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta &= 0\end{aligned}$$

projected on the **synchronization pattern**

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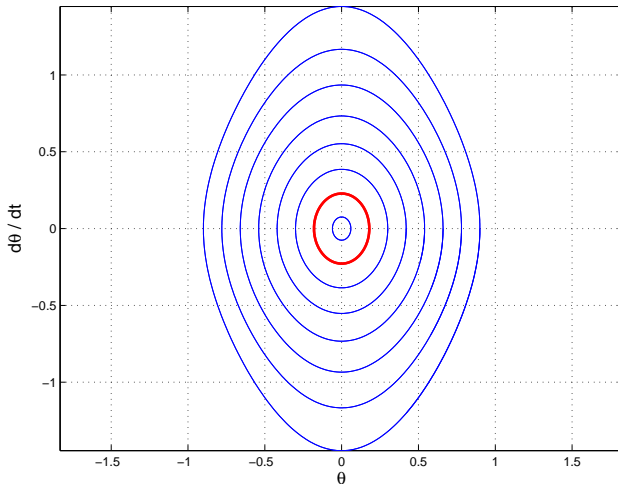
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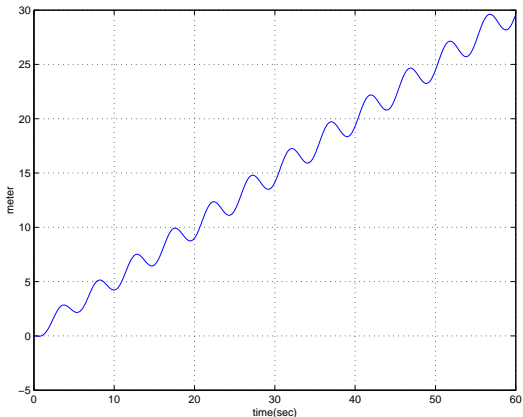
$$\ddot{\theta} + \dots + \omega^2 \sin \theta = 0$$

# Planning oscillations of a cart-pendulum system



The phase portrait of the reduced dynamics when  $\omega = 2\pi/5$  [rad/s]

# Planning a behavior of a cart-pendulum system



The position of the cart for the chosen  $\theta^*(t)$  behavior with

$$x^*(t) = - \left[ 1 + \frac{g}{\omega^2} \right] \ln \left( \frac{1 + \sin \theta^*(t)}{\cos \theta^*(t)} \right) + 0.5 \cdot t$$