Lecture 5

Orbital Stability and Stabilization for Underactuated Systems

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Stockholm, 2018-10-01

Learning outcomes: Concepts of moving Ponicare sections, transverse coordinates and transverse linearization for a solution of a nonlinear system. Andronov-Vitt theorem. Choices of transverse coordinates and transverse linarization for motions of underactuated mechanical systems. Examples

Outline

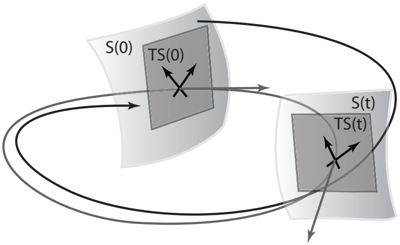
Transverse dynamics

Dynamics in a vicinity of a cycle

Given a T-periodic solution $x^*(\cdot)$, $x^*(t) = x^*(t+T)\ \forall\, t$, of the system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^{2n},$$

for analyzing its local properties introduce a family of transverse sections $\{S(\tau)\}$, $\tau \in [0, T]$, which union can recreate a tubing vicinity of the cycle.



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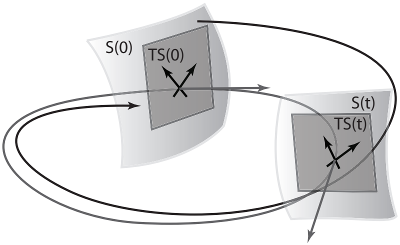

Defining hypersurfaces $\{S(\tau)\}_{\tau\in[0,T]}$ implies a change of coordinates:

$$\mathbb{R}^{2n} \ni x(\tau) \mapsto \left[\theta(\tau) \in \mathbb{R}^1, x_\perp \in S(\tau)\right], \qquad \tau \in [0, T]$$

such that in new coordinates the periodic solution $x^*(\cdot)$ is

$$\theta = \theta^*(t), \quad x_\perp^*(t) \equiv 0, \qquad \forall\, t$$

Orbital asymptotic stability of $x^*(\cdot)$ means that $x_\perp(t) \to 0$ as $t \to \infty$.

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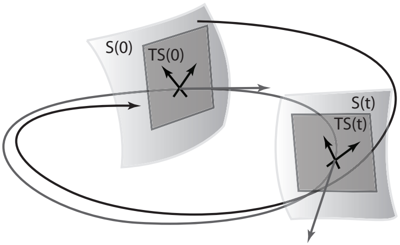

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The linearization of the dynamics of the system in a vicinity of $x^*(\cdot)$

$$\dot{z} = \left[\frac{\partial}{\partial x} f(x)\right]\Big|_{x=x^*(t)} z = A(t)z, \quad z \in \mathbb{R}^{2n}$$

- cannot be stable since $\theta(t) \not\to 0$ as $t \to +\infty$

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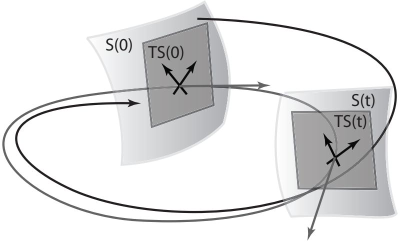

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- cannot be stable since $\theta(t) \not\to 0$ as $t \to +\infty$
- has a linear invariant subspace of co-dimension 1, which vectors approximate time evolution of transverse coordinates $x_\perp(\cdot)$.

Andronov-Vitt theorem (1930)

Given a T-periodic solution

$$x^*(t) = x^*(t+T) \quad \forall\, t$$

of the system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^{2n}.$$

Consider the linearization of the dynamics of the system in a vicinity of $x^*(\cdot)$

$$\begin{aligned} \dot{z} &= \left[\tfrac{\partial}{\partial x} f(x)\right]\Big|_{x=x^*(t)} z \\ &= A(t) z, \qquad z \in \mathbb{R}^{2n}. \end{aligned}$$

Consider the $(2n \times 2n)$-matrix function $\Phi(\cdot)$ defined as a solution of

$$\tfrac{d}{dt}\Phi(t) = A(t)\Phi(t), \qquad \Phi(0) = I_{2n}$$

One of eigenvalues of the monodromy matrix $\Phi(T)$ equals to 1.

If the amplitudes of $(2n-1)$ eigenvalues of the matrix $\Phi(T)$ are less than 1, then the solution $x^*(\cdot)$ is exponentially orbitally stable.

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Challenges in orbital feedback stabilization

Given a T-periodic solution $x^*(\cdot)$ of the nonlinear control system

$$\dot{x} = f(x) + g(x)u, \qquad x \in \mathbb{R}^{2n}, \; u \in \mathbb{R}^m,$$

obtained in response of an input $u^*(\cdot) \equiv 0$, consider the task to design a controller for asymptotic orbital stabilization of $x^*(\cdot)$

Any orbitally stabilizing feedback controller

$$u = U(x)$$

will satisfy the interpolation condition

$$U(x)\big|_{x=x^*(t)} \equiv 0, \quad \forall\, t$$

For a smooth $U(\cdot)$, it can be written as (see Hadamard lemma)

$$u = K(x)x_\perp(x),$$

where $x_\perp(\cdot)$ are transverse coordinates for the motion $x^*(\cdot)$

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In a vicinity of the T-periodic solution $x^*(\cdot)$ the system can be written in new coordinates

$$\tilde{x} = [\theta;\, x_\perp], \quad \theta \in \mathbb{R}^1,\ x_\perp \in \mathbb{R}^{2n-1}$$

as

$$\tfrac{d}{dt}\tilde{x} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})u.$$

With a feedback controller candidate

$$u = K(\tilde{x})x_\perp$$

the closed loop system becomes

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The linearization of the closed loop system in a vicinity of $x^*(\cdot)$ is

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By Andronov-Vitt theorem the linear feedback controller with the gain $k(\cdot)$ is not stabilizing the origin of this system

Challenges in orbital feedback stabilization

Orbital stabilization of the T-periodic solution $x^*(\cdot)$ the system

$$x \in \mathbb{R}^{2n}: \quad \tfrac{d}{dt}x = f(x) + g(x)u, \quad [x^*(\cdot),\, u^*(\cdot)]$$

by linearization requires feedback stabilization of **linearization** of dynamics of transverse coordinates

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With change of coordinates $x \mapsto \tilde{x} = [\theta \in \mathbb{R}^1;\, x_\perp \in \mathbb{R}^{2n-1}]$:

$$\tfrac{d}{dt}\tilde{x} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})u, \quad [\theta^*(\cdot);\, x_\perp^*(\cdot) \equiv 0;\, u^*(\cdot)]$$

It requires stabilizing an invariant subspace of the linearization

$$\dot{z} = A(t)z + B(t)v, \quad v = k(t)\delta x_\perp, \quad z \in \mathbb{R}^{2n},\, \delta x_\perp \in \mathbb{R}^{2n-1},\, v \in \mathbb{R}^n$$

of the system dynamics of co-dimension one: $\delta x_\perp \equiv 0$.

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of the system dynamics of co-dimension one: $\delta x_\perp \equiv 0$.

Generic choice of transverse coordinates

Given a T-periodic pair $[\, x^*(\cdot), u^*(\cdot) \,]$ of the system

$$\frac{d}{dt}x = f(x) + g(x)u, \quad x \in \mathbb{R}^{2n},\ u \in \mathbb{R}^m$$

how to define the transverse coordinates $x_\perp \in \mathbb{R}^{2n-1}$ for the motion?

Step 1: Given $t_\bullet \in [0, T]$ and the vector $v_0 := \frac{d}{dt}x^*(t_\bullet)$, choose a set of vectors

$$v_1 \in \mathbb{R}^{2n}, \quad v_2 \in \mathbb{R}^{2n}, \quad \ldots, \quad v_{2n-1} \in \mathbb{R}^{2n}$$

such that the $(2n \times 2n)$-matrix

$$[v_0, v_1, v_2, \ldots, v_{2n-1}]$$

is of rank $2n$

Step 2: Choose $v_i(t)$ for all $t \in [0, T]$ to be periodic and smooth.

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Transverse coordinates

Summary for Motion Representation of MS

Given an Euler-Lagrange system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = B(q)u$$

with

$$\dim q - \dim u = 1,$$

consider the geometrical relations

$$q_1 = \phi_1(\theta), \quad q_2 = \phi_2(\theta), \quad \ldots, \quad q_n = \phi_n(\theta)$$

relating the coordinates q_i and the new variable θ.

Compute the coefficients of the system

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0,$$

and choose one of its solutions, $\theta^\star(t)$.

Then, mechanical system has the motion

$$q_1(t) = \phi_1(\theta^\star(t)),\ q_2(t) = \phi_2(\theta^\star(t)),\ \ldots,\ q_n(t) = \phi_n(\theta^\star(t))$$

Summary for Motion Representation of MS

Given an Euler-Lagrange system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = B(q)u$$

with

$$\dim q - \dim u = 1,$$

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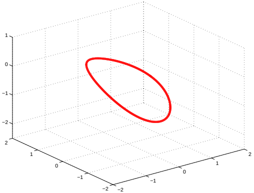
Transverse Coordinates for MS:



Given a T-periodic motion

$$q^\star(t) = \Big(q_1^\star(t),\, q_2^\star(t),\, \ldots,\, q_n^\star(t)\Big)^T$$

There are n-functions

$$\phi_1(\cdot),\ \phi_2(\cdot),\ \ldots,\ \phi_n(\cdot)$$

We can always assume that $q_n = \theta \quad \Rightarrow \quad \phi_n(\cdot)$ is trivial!

New n-generalized coordinates are θ and $y = \Big(y_1,\, \ldots,\, y_{n-1}\Big)$

$$y_1 = q_1 - \phi_1(\theta), \quad \ldots, \quad y_{n-1} = q_{n-1} - \phi_{n-1}(\theta)$$

For a motion of the E-L system with $x = (q_1, \ldots, q_n, \dot q_1, \ldots, \dot q_n)^T$ a candidate for transverse coordinates $x_\perp$ is given by

$$x_\perp = \Big[I(\theta, \dot\theta, \theta^\star(0), \dot\theta^\star(0)),\, y_1,\, \ldots,\, y_{n-1},\, \dot y_1,\, \ldots,\, \dot y_{n-1}\Big]^T$$

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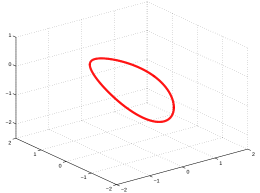

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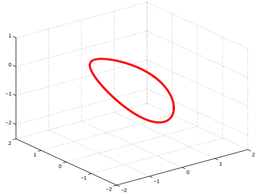

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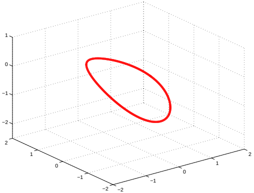

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Transverse linearization

Transverse Linearization: Example

Given a T-periodic motion $q^\star(t) = \left(q_1^\star(t),\, q_2^\star(t),\, \ldots,\, q_n^\star(t)\right)^T$ of the mechanical system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = B(q)u,$$

Rewrite the dynamics in the coordinates: θ, $y = (y_1, \ldots, y_{n-1})$

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Under certain assumptions one can define feedback transform

$$u = (u_1, \ldots, u_{n-1}) \to v = (v_1, \ldots, v_{n-1})$$

such that the dynamics are

$$\ddot{y} = v, \qquad \ddot{\theta} = N(\theta, \dot{\theta}, y, \dot{y}, \ddot{y})$$

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$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

Therefore the equation for $\ddot{\theta}$ can be rewritten as

$$\begin{aligned}\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) &= g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})y + \\ &\quad + g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\dot{y} + g_v(\theta, \dot{\theta}, y, \dot{y})\ddot{y}\end{aligned}$$

The functions $g_y(\cdot)$, $g_{\dot{y}}(\cdot)$ and $g_v(\cdot)$ are computed based on

- the Lagrangian $\mathcal{L}(\cdot)$ of the system;
- functions $\phi_1(\cdot), \ldots, \phi_{n-1}(\cdot)$ representing the target cycle

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The mechanical system written in y, θ-coordinates is

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\ddot{y} &= v
\end{aligned}$$

We search for linearization of transverse dynamics

$$x_\perp = \left[\, I(\theta, \dot{\theta}, \theta^\star(0), \dot{\theta}^\star(0));\ y;\ \dot{y} \,\right]$$

around the target solution: $y(t) \equiv 0$, $\theta(t) = \theta^\star(t)$

To do so, we use the differential relation

$$\begin{aligned}
\frac{d}{dt} I(\theta, \dot{\theta}, a, b) &= \dot{\theta}\left\{ \frac{2}{\alpha(\theta)} W - \frac{2\beta(\theta)}{\alpha(\theta)} I(\theta, \dot{\theta}, a, b) \right\} \\
&= \dot{\theta}\left\{ \frac{2}{\alpha(\theta)} \left[g_y(\cdot)y + g_{\dot{y}}(\cdot)\dot{y} + g_v(\cdot)v \right] - \frac{2\beta(\theta)}{\alpha(\theta)} I(\theta, \dot{\theta}, a, b) \right\}
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Transverse Linearization: Example

Consider the linear system of dimension $(2n-1)$

$$
\begin{aligned}
\frac{d}{dt} I_{\heartsuit}(t) &= \frac{2\,\dot{\theta}^{\star}(t)}{\alpha(\theta^{\star}(t))}\Big\{\Big[\tilde{g}_y(t) y_{\heartsuit}(t) + \tilde{g}_{\dot{y}}(t)\dot{y}_{\heartsuit}(t) + \tilde{g}_v(t) v_{\heartsuit}(t)\Big] - \\
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\ddot y_{\heartsuit}(t) &= v_{\heartsuit}(t)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde g_y(t) &= g_y(\theta, \dot\theta, \ddot\theta, y, \dot y)\Big|_{\substack{\theta=\theta^\star(t),\ \dot\theta=\dot\theta^\star(t) \\ \ddot\theta=\ddot\theta^\star(t),\ y=\dot y=0}} \\
\tilde g_{\dot y}(t) &= g_{\dot y}(\theta, \dot\theta, \ddot\theta, y, \dot y)\Big|_{\substack{\theta=\theta^\star(t),\ \dot\theta=\dot\theta^\star(t) \\ \ddot\theta=\ddot\theta^\star(t),\ y=\dot y=0}} \\
\tilde g_v(t) &= g_v(\theta, \dot\theta, y, \dot y)\Big|_{\substack{\theta=\theta^\star(t),\ \dot\theta=\dot\theta^\star(t) \\ y=\dot y=0}}
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If there is the stabilizing controller for this linear system

$$
v_\heartsuit(t) = K(t)\begin{bmatrix} I_\heartsuit \\ y_\heartsuit \\ \dot{y}_\heartsuit \end{bmatrix}, \quad K(t) = K(t+T),
$$

then we suggest a nonlinear controller that stabilizes orbitally the target periodic motion for the mechanical system