Lecture 6

Orbital Stability and Stabilization for Hybrid Systems

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Stockholm, 2018-10-05

Learning outcomes: Hybrid dynamics and motion planning. Hybrid transverse linearization for a solution of a mechanical system. Examples

Outline

1. Problem formulation and examples
    - Gaits of a passive compass biped
    - Perpetual rotations of a devil stick
2. Planning a cycle of a hybrid mechanical system
3. Orbital stabilization of a hybrid cycle
    - Choices for moving Poincare sections
    - Hybrid transverse linearization
4. Analysis of gaits of a passive compass biped

Problem formulation

Problem formulation

Given

- Controlled mechanical system

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = B(q)u,$$

with $q = [q_1, \ldots, q_n]$, $u = [u_1, \ldots, u_m]$

Tasks:

- To plan a hybrid motion, i.e. a pair $[q_\star(\cdot), u_\star(\cdot)]$
- To design a controller to achieve contraction to the orbit of the motion $q_\star(\cdot)$

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- Discrete dynamics (instantaneous maps)

$$F^{(i)} : \Gamma_{-}^{(i)} \to \Gamma_{+}^{(i)} \quad i = 1, \ldots, N_d$$

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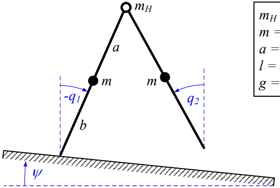
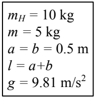
Gaits of a passive compass biped





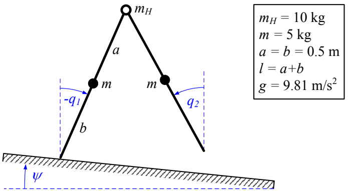
The continuous-in-time arc of the hybrid gait is the solution of Euler-Lagrange equations shaped by a zero control input:

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_1}\mathcal{L}\right] - \frac{\partial}{\partial q_1}\mathcal{L} = 0, \qquad \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_2}\mathcal{L}\right] - \frac{\partial}{\partial q_2}\mathcal{L} = 0,$$

where the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}^T \underbrace{\begin{bmatrix} p_1 & -p_2\cos(q_1-q_2)\\ -p_2\cos(q_1-q_2) & p_3\end{bmatrix}}_{=M(q)} \begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} - \underbrace{-\left[p_4\left(\cos(q_1)-1\right) + p_5\left(1-\cos(q_2)\right)\right]}_{=V(q)}$$

Gaits of a passive compass biped



The discrete dynamics appear due to impact activated when the continuous-in-time solution hits the surface

$$\Gamma_- = \{[q, \dot{q}] : \cos(q_1 + \psi) - \cos(q_2 + \psi) = 0\}$$

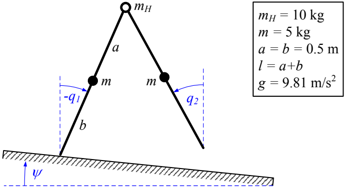
The discrete dynamics change the state instantaneously by the rule

$$F : \Gamma_- \ni [q_-, \dot{q}_-] \mapsto [q_+, \dot{q}_+] \in \Gamma_+$$

where

$$q_+ = F_1 q_-, \quad \dot{q}_+ = F_2(q_-)\dot{q}_- \qquad \Gamma_+ = \Gamma_-$$

Gaits of a passive compass biped



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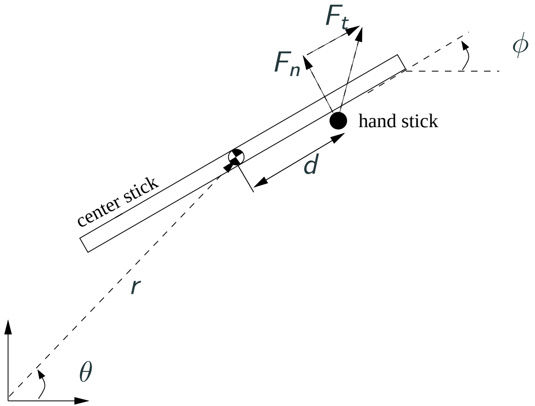
The task is to find a periodic gait and analyze its stability

Perpetual rotations of a devil stick



Devil stick consists of a (rotating) **centre stick** driven by a force generated by a **hand stick**, which in turn rolls on the boundary of centre stick without slipping

Perpetual rotations of a devil stick

The dynamics of the centre stick written in polar coordinates are

$$
\begin{aligned}
\ddot{r} &= r\dot{\theta}^2 - g\sin\theta + \frac{\cos(\theta-\phi)}{m}F_t + \frac{\sin(\theta-\phi)}{m}F_n \\
\ddot{\theta} &= -\frac{2\dot{r}\dot{\theta}}{r} - \frac{g\cos\theta}{r} - \frac{\sin(\theta-\phi)}{rm}F_t + \frac{\cos(\theta-\phi)}{rm}F_n \\
\ddot{\phi} &= \frac{d(\phi)}{J}F_n = \frac{-\rho\phi + d_0}{J}F_n
\end{aligned}
$$

- (r, θ) are coordinates for center of mass of centre stick;
- ϕ is the angle that center stick makes with horizontal;
- $d(\phi)$ is the instantaneous position at which the centre stick and the hand stick are in contact $d(\phi) = -\rho\phi + d_0$;
- d_0 is the initial contact position when $\phi = 0$;
- ρ is the radius of the hand stick, m is the mass of the centre stick, J is its moment of inertia;
- F_t and F_n are tangential and normal components of force.

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Searching for a propeller motion for the centre stick

The task are to **create** a propeller motion by $F_t(\cdot)$, $F_n(\cdot)$

- being "periodic" in coordinates

$$\phi(T) = \phi(0) + 2\pi, \quad \theta(T) = \theta(0) + 2\pi, \quad r(T) = r(0)$$

with $d(\phi) = -\rho\phi + d_0$, where T is some positive constant;

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- being periodic in velocities, i.e.

$$\lim_{\varepsilon \to 0_+} \tfrac{d}{dt}\phi(T - \varepsilon) = \lim_{\varepsilon \to 0_+} \tfrac{d}{dt}\phi(T + \varepsilon)$$

  even at $t = T$, where the hand and centre sticks change instantaneously the contact point in-between to its original value at $t = 0$.

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and to **stabilize** the hybrid cycle.

Origin of discrete dynamics

For both examples, the discrete dynamics are necessary for the description of the nominal motion

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But the reasons the update of the state is included in the description are different

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- For the devil stick, the update is due to the control strategy

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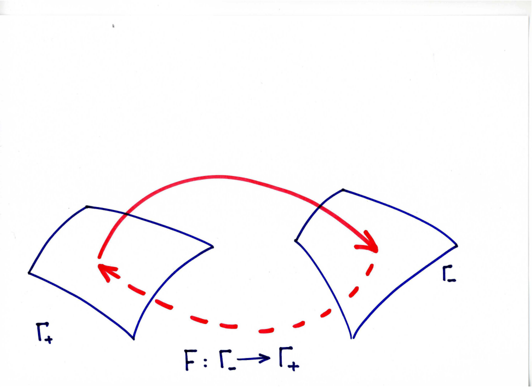
But the reasons the update of the state is included in the description are different

- For the compass biped the update is due to the physics of the impact forces
- For the devil stick, the update is due to the control strategy
- Mechanical systems can instantly change a number of degrees of freedom!

Planning a hybrid cycle

Hybrid cycle with one jump



A cycle consists of a sub-arc of the continuous-in-time motion and the update map, which brings the end point to the point where the cycle starts

Hybrid cycle with one jump

Assume that a motion of a mechanical system with n degrees of freedom

$$q_1 = q_1(t), \quad q_2 = q_2(t), \quad \ldots, \quad q_n = q_n(t), \quad t \in [0, T]$$

can be represented as

$$q_1(t) = \phi_1(\theta(t)), \quad q_2(t) = \phi_2(\theta(t)), \quad \ldots, \quad q_n(t) = \phi_n(\theta(t)),$$

where the scalar variable

$$\theta = \theta(t), \quad t \in [0, T]$$

is a generator of the motion.

If the system is underactuated then $\theta(\cdot)$ is a solution of

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

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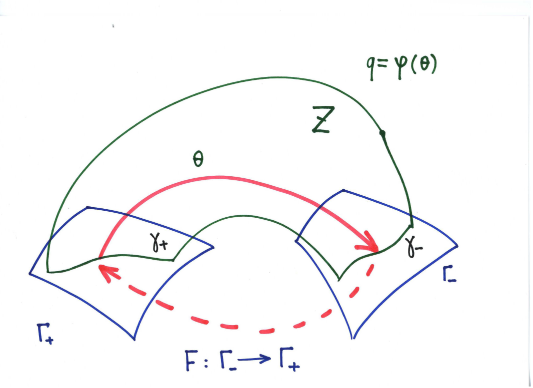
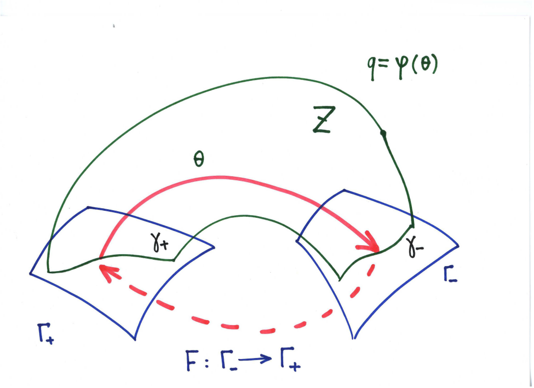


The relations $q = \phi(\theta)$ associated with the motion define a 2-D surface Z in the state space of a robot. The curves γ_+ and γ_- are the intersections of Z with Γ_+ and Γ_- respectively.

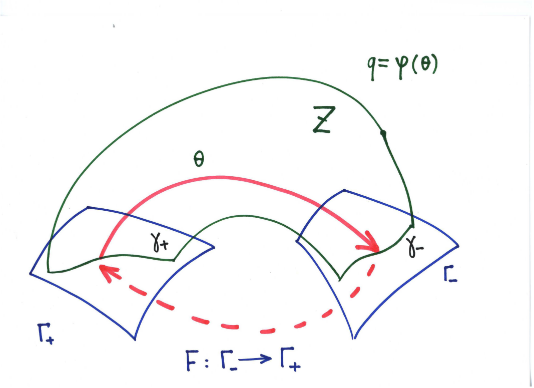
Hybrid cycle with one jump



Behavior of $\theta(\cdot)$ along the motion is not any, it is the solution of

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

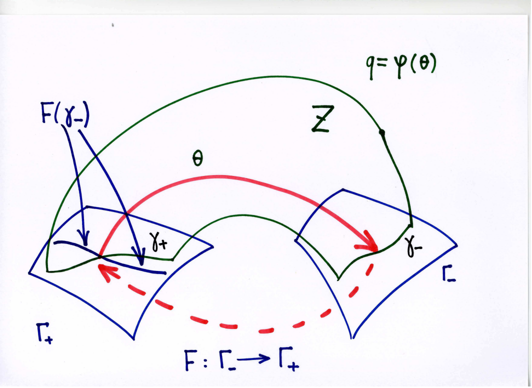
Hybrid cycle with one jump



Integrability $\Rightarrow$ $[\theta(T_h), \dot\theta(T_h)]$ can be found by solving Eqns:

$$I\left([\theta(T_h), \dot\theta(T_h)], [\theta(0), \dot\theta(0)]\right) = 0, \quad \left[q(\theta(T_h)), \dot q(\theta(T_h))\right] \in \Gamma_-$$

Hybrid cycle with one jump



The discrete dynamics of the hybrid mechanical system do not necessary keep the 2-D manifold Z invariant. In general,

$$F(\gamma_-) \not\subset \gamma_+$$

Hybrid cycle with one jump

- Choose a parametric set of C^2-smooth functions

$$\phi(\theta, P) = \{\phi_1(\theta, P), \, \phi_2(\theta, P), \, \ldots, \, \phi_n(\theta, P)\}$$

Hybrid cycle with one jump

- Choose a parametric set of C^2-smooth functions

$$\phi(\theta, P) = \{\phi_1(\theta, P),\ \phi_2(\theta, P),\ \ldots,\ \phi_n(\theta, P)\}$$

- Compute 2-D manifolds $Z(P)$ and projected dynamics

$$\alpha(\theta, P)\ddot{\theta} + \beta(\theta, P)\dot{\theta}^2 + \gamma(\theta, P) = 0$$

Any solution $\theta(t, P)$, if exists till $t = \tau_1 > 0$, satisfies

$$I\left([\theta(\tau_1, P), \dot{\theta}(\tau_1, P)], [\theta(0, P), \dot{\theta}(0, P)]\right) = 0$$

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- Compute curves: $\gamma_+ = \Gamma_+ \cap Z(P)$, $\gamma_- = \Gamma_- \cap Z(P)$ and

$$\mathcal{F}\left([\theta, \dot{\theta}]\right)\Big|_{[\theta,\dot{\theta}]\in\gamma_-} = F\left([q, \dot{q}]\right)\Big|_{\substack{q=\phi(\theta,P) \\ \dot{q}=\phi'(\theta,P)\,\dot{\theta}}}$$

Hybrid cycle with one jump

- $\phi(\theta, P) = \{\phi_1(\theta, P),\ \phi_2(\theta, P),\ \ldots,\ \phi_n(\theta, P)\}$
- Compute 2-D manifolds $Z(P)$ and the integral

$$I\left([\theta(\cdot, P), \dot{\theta}(\cdot, P)], [\theta(0, P), \dot{\theta}(0, P)]\right) = 0$$

- Compute $\gamma_+(P)$, $\gamma_-(P)$ and $\mathcal{F}([\theta, \dot{\theta}], P)$

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Organize a search for

parameters $P = P_\star$ and the scalars – a, b, x, y

that solve the equations

$$I\left([a, b], [x, y], P\right) = 0 \qquad \mathcal{F}\left([a, b], P\right) = [x, y]$$

$$[a, b] \in \gamma_-(P) \qquad [x, y] \in \gamma_+(P)$$

Hybrid cycle with one jump

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Hybrid cycle with one jump

Consider a system that lives in $[\theta, \dot{\theta}]$-plane with $\theta \geq 0$: its continuous dynamics are defined for $\theta > 0$ as

$$\ddot{\theta} + \theta = 0$$

and its instantaneous law is defined on $\theta = 0$ as

$$\theta_+ = 0, \quad \dot{\theta}_+ = (\dot{\theta}_-) \cdot \sin(\dot{\theta}_-)$$

To find periodic motions for this hybrid system, one needs to solve the algebraic equations $\theta_+ = 0$, $\theta_- = 0$ and

$$\begin{aligned}(\dot{\theta}_+)^2 + (\theta_+)^2 - (\dot{\theta}_-)^2 - (\theta_-)^2 &= 0 \\ \dot{\theta}_+ &= (\dot{\theta}_-) \cdot \sin(\dot{\theta}_-)\end{aligned}$$

Solutions are

$$\theta_+ = 0, \quad \dot{\theta}_+ = 2\pi \cdot n + \frac{\pi}{2}, \quad n = 0, 1, \ldots$$

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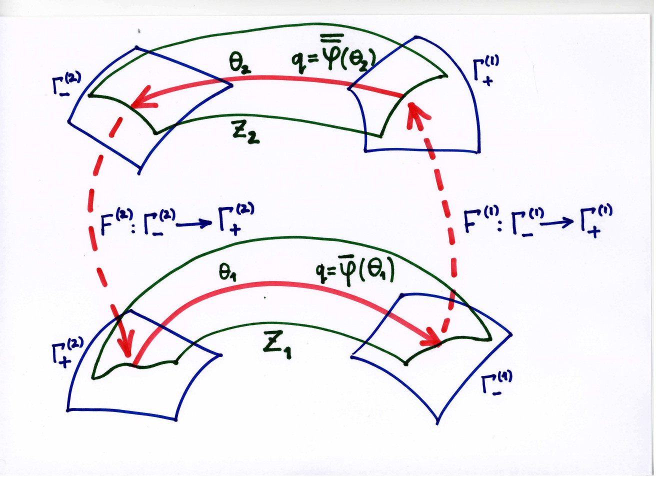
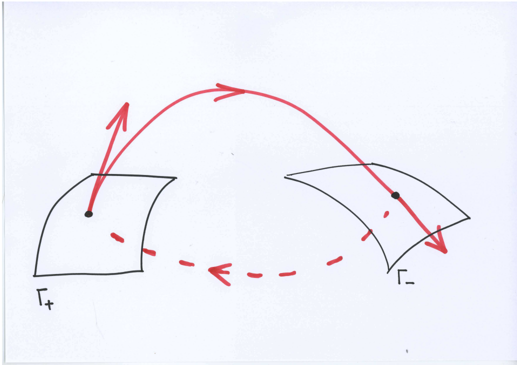


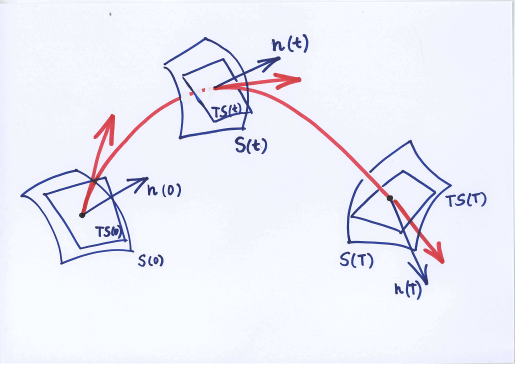
Similar constructions for a hybrid cycle with two jumps

Orbital stabilization of a hybrid cycle



A hybrid cycle with one jump

Hybrid cycle with one jump



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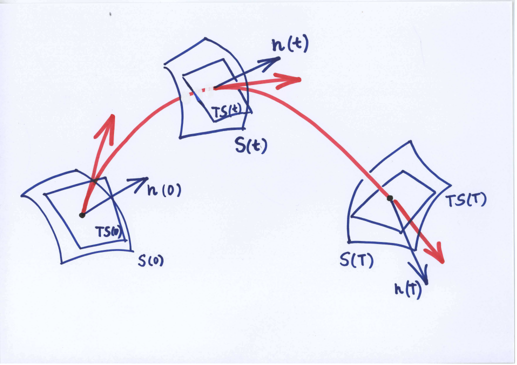
Hybrid cycle with one jump



- Moving Poincaré section is a collection of $(2n-1)$-D surfaces $S(t)$ transversal to the continuous-in-time sub-arc;
- The state $[q, \dot{q}] \in \mathbb{R}^n \times \mathbb{R}^n$ can be transformed into $[\theta, x_\perp] \in \mathbb{R}^1 \times \mathbb{R}^{2n-1}$. Here $\theta(t)$ defines point along the orbit, $x_\perp(t) \in S(t)$ are transverse coordinates.

Given a T-periodic motion

$$q^\star(t) = \Big(q_1^\star(t),\ q_2^\star(t),\ \ldots,\ q_n^\star(t)\Big)^T$$

There are a motion generator θ; n-functions for its nested representation

$$\phi_1(\cdot),\ \phi_2(\cdot),\ \ldots,\ \phi_n(\cdot)$$

We can always assume that one of them, i.e. $\phi_n(\cdot)$, is trivial!

New n-generalized coordinates are θ and $y = \Big(y_1,\ \ldots,\ y_{n-1}\Big)$

$$y_1 = q_1 - \phi_1(\theta), \quad \ldots, \quad y_{n-1} = q_{n-1} - \phi_{n-1}(\theta)$$

For a motion of the E-L system with $x = (q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)^T$ a candidate for transverse coordinates $x_\perp$ is given by

$$x_\perp = \Big[I(\theta, \dot{\theta}, \theta^\star(0), \dot{\theta}^\star(0)),\ y_1,\ \ldots,\ y_{n-1},\ \dot{y}_1,\ \ldots,\ \dot{y}_{n-1}\Big]^T$$

Transverse Coordinates for MS:

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$$y_1 = q_1 - \phi_1(\theta), \quad \ldots, \quad y_{n-1} = q_{n-1} - \phi_{n-1}(\theta)$$

For a motion of the E-L system with $x = (q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)^T$ a candidate for transverse coordinates $x_\perp$ is given by

$$x_\perp = \left[I(\theta, \dot{\theta}, \theta^{\star}(0), \dot{\theta}^{\star}(0)),\, y_1,\, \ldots,\, y_{n-1},\, \dot{y}_1,\, \ldots,\, \dot{y}_{n-1}\right]^T$$

Transverse Coordinates for MS:

Given a T-periodic motion

$$q^\star(t) = \Big(q_1^\star(t),\, q_2^\star(t),\, \ldots,\, q_n^\star(t)\Big)^T$$

There are a motion generator θ; n-functions for its nested representation

$$\phi_1(\cdot),\ \phi_2(\cdot),\ \ldots,\ \phi_n(\cdot)$$

We can always assume that one of them, i.e. $\phi_n(\cdot)$, is trivial!

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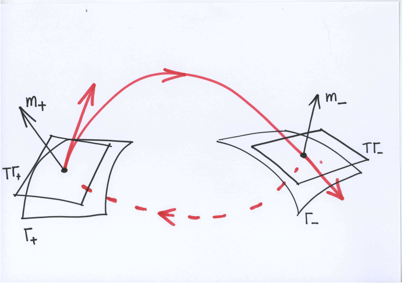
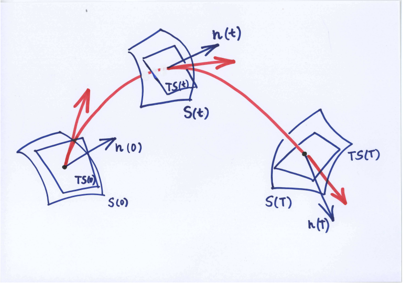
Hybrid transverse linearization



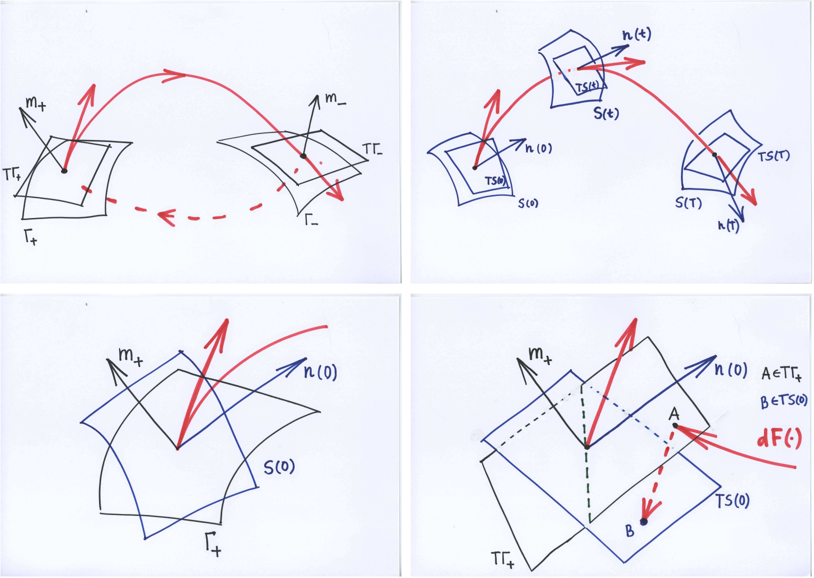


Two linear mappings

- Linearization of the update law $F(\cdot)$
- Linearization of the dynamics of transverse coordinates $x_\perp$

should be combined!

Hybrid transverse linearization



Hybrid transverse linearization

Let $q_\star(t) = q_\star(t+T)$, $u_\star(t) = u_\star(t+T)$ be a hybrid cycle of

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q,\dot{q})u, \qquad F : \Gamma_- \to \Gamma_+$$

Introduce a linear hybrid system defined by

- linear continuous time system

$$\frac{d}{d\tau} z = A\left(\tau \bmod T\right) z + B\left(\tau \bmod T\right) v$$

activated on time intervals

$$[0, T],\ [T, 2T],\ \ldots,\ [kT, (k+1)T],\ \ldots$$

It is responsible for orbital stability/stabilization of a cycle

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$$z_- \quad \mapsto \quad z_+ = L z_-$$

  that re-initializes the state at times T, $2T$, $\ldots$, kT, $\ldots$

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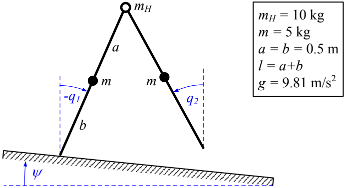
Analysis of gaits of a passive compass biped

Gaits of a passive compass biped



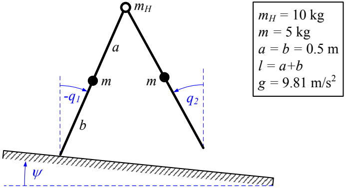
The continuous-in-time arc of the hybrid gait is the solution of Euler-Lagrange equations shaped by a zero control input:

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_1}\mathcal{L}\right] - \frac{\partial}{\partial q_1}\mathcal{L} = 0, \qquad \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_2}\mathcal{L}\right] - \frac{\partial}{\partial q_2}\mathcal{L} = 0,$$

where the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}^T \underbrace{\begin{bmatrix} p_1 & -p_2\cos(q_1 - q_2) \\ -p_2\cos(q_1 - q_2) & p_3 \end{bmatrix}}_{= M(q)} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} - \underbrace{\left[p_4\left(\cos(q_1) - 1\right) + p_5\left(1 - \cos(q_2)\right)\right]}_{= V(q)}$$

Gaits of a passive compass biped



The discrete dynamics appear due to impact activated when the continuous-in-time solution hits the surface

$$\Gamma_- = \{[q, \dot{q}] : \cos(q_1 + \psi) - \cos(q_2 + \psi) = 0\}$$

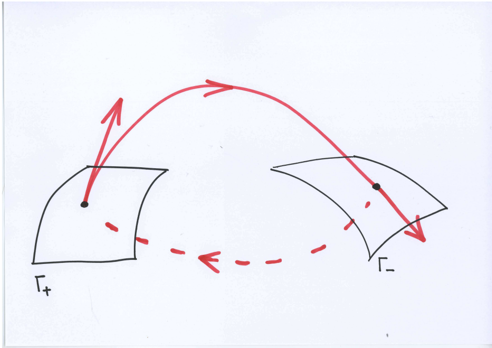
The discrete dynamics change the state instantaneously by the rule

$$F : \Gamma_- \ni [q_-, \dot{q}_-] \mapsto [q_+, \dot{q}_+] \in \Gamma_+$$

where

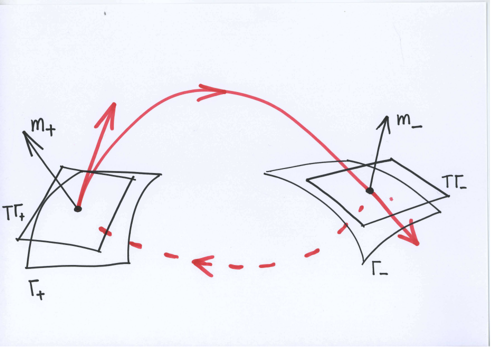
$$q_+ = F_1 q_-, \quad \dot{q}_+ = F_2(q_-)\dot{q}_- \qquad \Gamma_+ = \Gamma_-$$

Symmetric gait of a compass biped



- A cycle consists of sub-arc of continuous-in-time motion and the update map of the end point to the beginning.
- One of hypersurfaces Γ_-, Γ_+ is commonly used as Poicare section for analyzing the orbital stability of the gait.

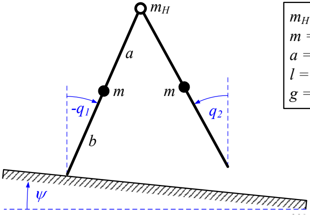
Symmetric gait of a compass biped



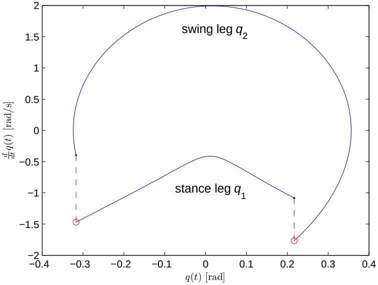
Tangent planes $T\Gamma_-$ and $T\Gamma_+$ of Γ_- and Γ_+ at two points of the cycle are used for defining the (numerical) linearization of Poincare map:

$$z_{k+1} = Az_k, \quad z_{k+1}, z_k \in T\Gamma_+, \quad \dim T\Gamma_+ = (2n-1) = 3$$

A stable gait for $\psi = 2.87^\circ$

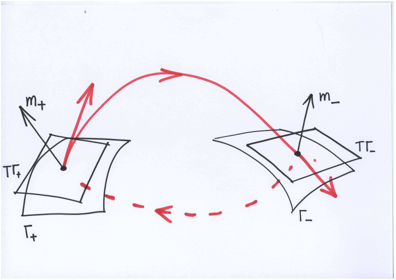


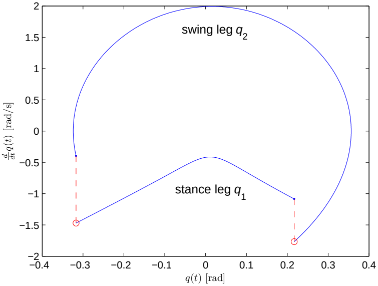
$m_H = 10$ kg
$m = 5$ kg
$a = b = 0.5$ m
$l = a+b$
$g = 9.81$ m/s^2



One of stable cycles of the walker shown
on the combined phase portrait

A stable gait for $\psi = 2.87°$





$$\vec{m}_+ = \left[-\sin(q_{1\star}(0) + \psi), 0, \sin(q_{2\star}(0) + \psi), 0\right]^T \approx \left[-0.26, 0, 0.26.0\right]^T$$

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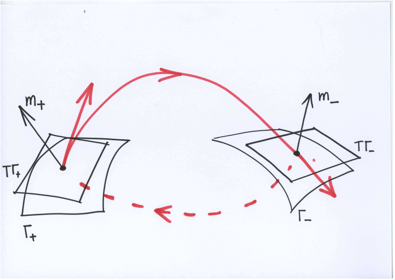

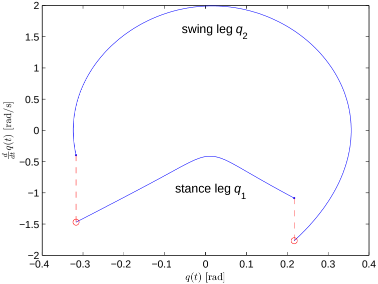

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$$\vec{f}_\star(0) = \left[\dot{q}_{1\star}(0), \ddot{q}_{1\star}(0), \dot{q}_{2\star}(0), \ddot{q}_{2\star}(0)\right] \approx [-1.08, 3.8, -0.39, 11.47]$$

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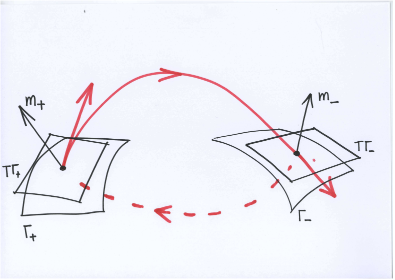

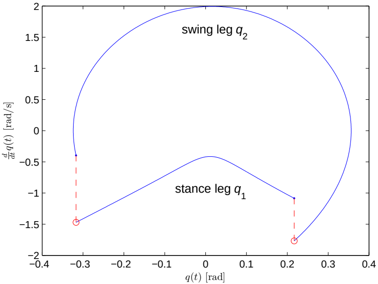

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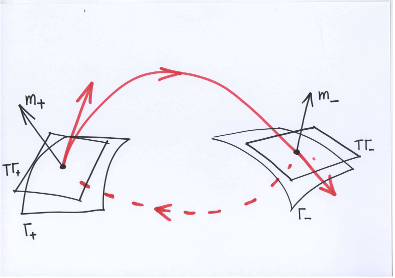
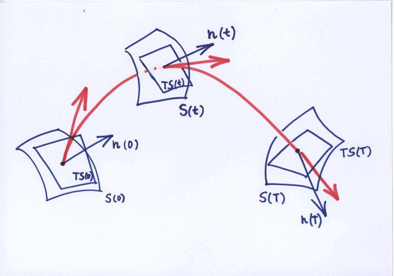
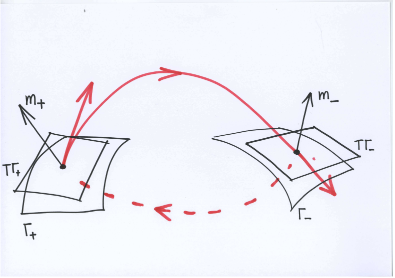
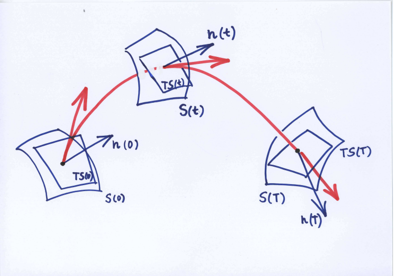
$$\cos\left(\widehat{\vec{m}_+, \vec{f}_\star(0)}\right) = \frac{\vec{m}_+ \cdot \vec{f}_\star(0)}{|\vec{m}_+||\vec{f}_\star(0)|} \approx 0.18$$





There are plenty of other choices of transverse sections $S(t)$, $t \in [0, T_p]$, for continuous-in-time arc of the cycle.



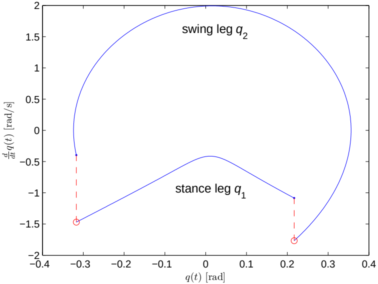


There are plenty of other choices of transverse sections $S(t)$, $t \in [0, T_p]$, for continuous-in-time arc of the cycle.

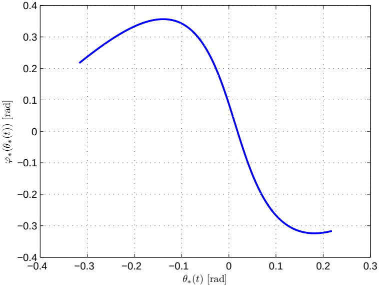
There are special Poincare surfaces $S(t)$: **orthogonal** to the vector field on the cycle at any time

$$\left\{ \vec{f}_\star(t) \right\} \perp \left\{ T S(t) \right\}, \qquad \forall\, t \in [0, T_p]$$

Symmetric gait and its nested representation



(a) Phase portrait



(b) The recomputed kinematic relation $q_2 = \phi(q_1)$ between the coordinates on the gait

Transverse coordinates for the gait

Given a motion

$$q_\star(t) = \left[q_{1\star}(t),\ q_{2\star}(t) \right], \quad t \in [0, T_e]$$

and associated virtual holonomic constraint

$$q_{2\star}(t) = \phi(q_{1\star}(t))$$

New generalized coordinates are θ and y with

$$\theta = q_1, \quad y = q_2 - \phi(\theta)$$

For the system with $x = (q_1, q_2, \dot{q}_1, \dot{q}_2)^T$, $\dim x = 4$, three transverse coordinates $x_\perp$ can be defined by

$$x_\perp = \left[I(\theta, \dot{\theta}, \theta_\star(0), \dot{\theta}_\star(0)),\ y,\ \dot{y} \right]^T$$

where $I(\cdot)$ is the integral of the (α, β, γ)-equation.

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Hybrid transverse linearization is defined by

- the linear continuous time system

$$\tfrac{d}{d\tau} z = A\left(\tau \bmod T\right) z + B\left(\tau \bmod T\right) v$$

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$$[0, T],\ [T, 2T],\ \ldots,\ [kT, (k+1)T],\ \ldots$$

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Hybrid transverse linearization for the compass biped gait

Given a cycle $q_\star(t) = q_\star(t + T_e)$ of the compass-gait walker,

- There is a linear hybrid system

$$\frac{d}{d\tau}z = A(\tau)z, \; z \in \mathbb{R}^3, \; A(\tau) = A(\tau + T_e), \qquad z_+ = L z_-$$

which solutions approximate (in some sense) solutions of the walker in a tubing vicinity of the orbit.

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- If it is integrated over period $T_e \approx 0.7324$ s, then we obtain

$$z_{k+1} = A_0 z_k, \quad k = 0, 1, 2, \ldots$$

where

$$A_0 = \begin{bmatrix} 0.106396835734031 & 3.885253845507885 & 0.395255324457403 \\ -0.114586152743185 & 0.668714216213521 & 0.062728332694704 \\ 0.488888548158536 & -7.764952916374034 & -0.673609328082623 \end{bmatrix}$$

It gives the linearization of the first-return Poincare map.