

# Lecture 7

## Motion Control of the Butterfly Robot

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**Learning outcomes:** Generalized Coordinates, Curve Parametrization, Lagrange Equations of the I-st kind, Mechanical Constraints, Motion Planning Problem

1. The Butterfly Robot Overview
2. Physics of the Butterfly Robot
3. Motion Planning

# The Butterfly Robot Overview

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# The Butterfly Robot Overview

- Study of the non-prehensile manipulation problem
- How to teach a robot to move an object with some internal dynamics?
- With non-trivial dynamics of objects interaction?



# The Butterfly Robot Overview

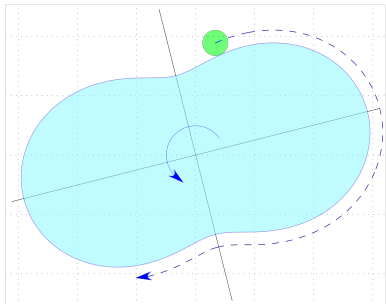
The problem depends significantly on physics of the interaction!

Physics of contact between rigid bodies.

The set-up was developed as a simplified version of the non-prehensile manipulation problem in 2D

This allows to study the problems:

- verification of rigid-body contact model
- slipping
- friction



# Sensors, Actuators

1. the transparent plates
2. DC servo drive, Maxon brushed motor with linear characteristic torque-current
3. high res encoder 8192pts/rev
4. industrial camera 200fps, 1.3Mpx
5. the CPU is processing the video stream, estimates position of the ball with accuracy  $\approx 20\mu\text{m}$ , by  $0.5\text{msec}$

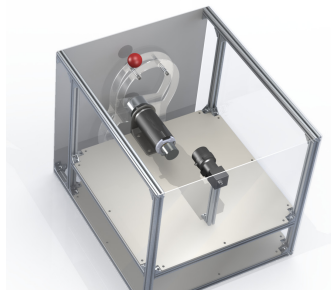
The signals are being measured:

- position of the ball:  $x$ ,  $y$ , and maybe  $\psi$
- position of the plates:  $\theta$

The control input:

- torque  $\tau$

CV will be considered on the next lec



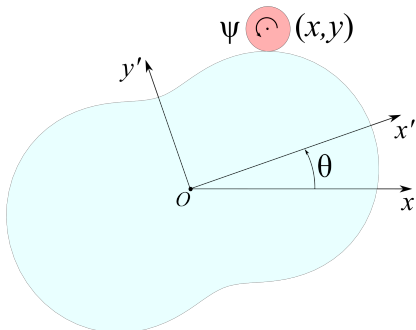
# Physics of the Butterfly Robot

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# Kinematics of the Robot

The “standard” generalized coordinates

- ball has 3DOF
- plates have 1DOF
- system has 4DOF
- 4 generalized coordinates
- $x, y$  – ball center
- $\psi$  – ball angular position
- $\theta$  – angular position of plates



**the coordinates do not contain  
information about the curve shape!**

## Geometry of a Plane Curve

We need to introduce the “local” coordinates which describe ball position in a vicinity of the plates!

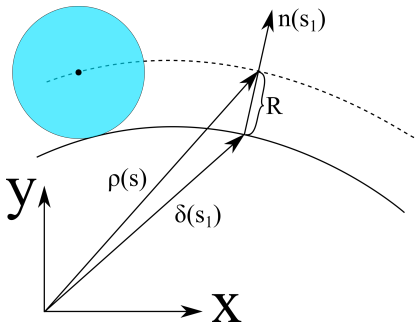
- Shape of the plates is given by  $\delta$
- Plates in parametric form:  
 $\delta : \mathbb{R} \rightarrow \mathbb{R}^2$
- Assume  $s_1$  is natural:  $\left\| \frac{d\delta(s_1)}{ds_1} \right\| = 1$
- define an auxiliary curve

$$\rho(s_1) = \delta(s_1) + n(s_1) \cdot R$$

$$\text{with } n(s_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \delta'(s_1)$$

- reparametrize the curve  $\rho$ :

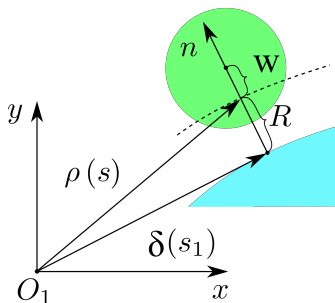
$$s(s_1) = \int^{s_1} \left\| \frac{d\rho}{ds_1} \right\| ds_1$$



## Geometry of a Plane Curve

New generalized coordinates:

- ball position along the curve  $\rho$  is given by  $s$
- in the perpendicular direction is given by  $w$
- it's angular position relative to the frame is given by  $\psi$



ball Cartesian coordinates:

$$r = T(\theta)(\rho(s) + n(s)w)$$

$$n(s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rho' = A\rho', \quad T(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

## System Energy

We have generalized coordinates  $q = (\theta, s, w, \psi)$

Kinetic energy of the system

$$K = \frac{1}{2} m \left\| \frac{d}{dt} r(\theta, w, s) \right\|^2 + \frac{1}{2} J_d (\dot{\psi} + \dot{\theta})^2 + \frac{1}{2} J_p \dot{\theta}^2$$

differentiate  $r$  over time:

$$\dot{r} = T' \cdot \dot{\theta} (\rho + nw) + T \cdot (\rho' \dot{s} + n' w \dot{s} + n \dot{w})$$

and substitute

$$\begin{aligned} \|\dot{r}\|^2 &= \dot{\theta}^2 \left( \|\rho\|^2 + w^2 + 2\rho^T n w \right) \\ &+ \left( 1 + \|k\|^2 w^2 + 2\tau \times k w \right) \dot{s}^2 + \dot{w}^2 \\ &+ 2 \left( \rho^T n + w - \rho^T k w + \tau \times k w^2 \right) \dot{s} \dot{\theta} - 2\rho^T \tau \dot{w} \dot{\theta} \end{aligned}$$

we have kinetic energy in a compact form

$$K(q, \dot{q}) = \dot{q}^T M(q) \dot{q}$$

with  $q = (\theta, s, w, \psi)$  and

$$M(q) = m \begin{pmatrix} \|\rho\|^2 + w^2 + 2\rho^T n w + \frac{J_p}{m} + \frac{J_d}{m} & \rho^T n + w - \rho^T k w + \tau \times k w^2 & -\rho^T \tau & \frac{J_d}{m} \\ \rho^T n + w - \rho^T k w + \tau \times k w^2 & 1 + \|k\|^2 w^2 + 2\tau \times k w & 0 & 0 \\ -\rho^T \tau & 0 & 1 & 0 \\ \frac{J_d}{m} & 0 & 0 & \frac{J_d}{m} \end{pmatrix}$$

Potential energy:

$$\begin{aligned} U(q) &= mg \cdot (0, 1) \cdot r \\ &= mg \cdot (0, 1) \cdot T(\theta) (\rho(s) + n(s)w) \end{aligned}$$

# Lagrange Equations

Lagrange equations:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = f$$

$$M(q) = m \begin{pmatrix} \|\rho\|^2 + w^2 + 2\rho^T n w + \frac{J_p}{m} + \frac{J_d}{m} & \rho^T n + w - \rho^T k w + \tau \times k w^2 & -\rho^T \tau & \frac{J_d}{m} \\ \rho^T n + w - \rho^T k w + \tau \times k w^2 & 1 + \|k\|^2 w^2 + 2\tau \times k w & 0 & 0 \\ -\rho^T \tau & 0 & 1 & 0 \\ \frac{J_d}{m} & 0 & 0 & \frac{J_d}{m} \end{pmatrix}$$

$$C(q, \dot{q}) = \dots$$

$$G(q) = mg \begin{pmatrix} (0, 1) \cdot T'(\rho + n w) \\ (0, 1) \cdot T(\tau + n' w) \\ (0, 1) \cdot T n \\ 0 \end{pmatrix}$$

## Mechanical Constraints

We are mostly interested in motion with  $w = 0$ :

$$m \begin{pmatrix} \|\rho\|^2 + \frac{J_p}{m} + \frac{J_d}{m} & \rho^T n & -\rho^T \tau & \frac{J_d}{m} \\ \rho^T n & 1 & 0 & 0 \\ -\rho^T \tau & 0 & 1 & 0 \\ \frac{J_d}{m} & 0 & 0 & \frac{J_d}{m} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{s} \\ 0 \\ \ddot{\psi} \end{pmatrix} +$$

$$m \begin{pmatrix} \rho^T \tau \dot{s} & \rho^T \tau \dot{\theta} + \rho \times k \dot{s} & \rho^T n \dot{\theta} - \rho^T k \dot{s} & 0 \\ -\rho^T \tau \dot{\theta} & 0 & \dot{\theta} + \tau \times k \dot{s} & 0 \\ -\rho^T n \dot{\theta} - \dot{s} & -\dot{\theta} - \tau \times k \dot{s} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{s} \\ 0 \\ \dot{\psi} \end{pmatrix} +$$

$$mg \begin{pmatrix} (0, 1)^T \rho \\ (0, 1)^T \tau \\ (0, 1)^T n \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ 0 \\ F_n \\ 0 \end{pmatrix}$$

Normal component of the reaction force

$$F_n = m \left( -\rho^T \tau \ddot{\theta} - \rho^T n \ddot{\theta}^2 - \tau \times k \dot{s}^2 - 2\dot{s} \dot{\theta} + g (0, 1)^T n \right)$$

or if plates are much more massive, i.e.  $\ddot{\theta} \approx u/J_p$

$$F_n \approx m \left( -\rho^T \tau \frac{u}{J_p} - \rho^T n \dot{\theta}^2 - \tau \times k \dot{s}^2 - 2\dot{s} \dot{\theta} + g \cdot (0, 1) \cdot n \right)$$

A disk is sliding over a rotating curve

$$\begin{aligned} & \left( \|\rho\|^2 + \frac{J_p}{m} + \frac{J_d}{m} \right) \ddot{\theta} + \rho^T n \ddot{s} + J_d \ddot{\psi} + \\ & \rho^T \tau \dot{\theta}^2 + \rho^T \tau \dot{\theta} \dot{s} + \rho \times k \dot{s}^2 + g(0, 1) T' \rho = u/m \\ & \rho^T n \ddot{\theta} + m \ddot{s} - \rho^T \tau \dot{\theta}^2 + g(0, 1) T \tau = 0 \\ & J_d (\ddot{\theta} + \ddot{\psi}) = 0 \end{aligned}$$

# Mechanical Constraints

No slipping: velocity of the contact point

$$\dot{\psi} = -\frac{1}{R_{ball}} \dot{s}$$

Holonomic constraint?

$$\psi = -\frac{s}{R_{ball}}$$

Holonomic constraint!

Lagrange Equations of the 1-st kind

$$\left( \|\rho\|^2 + \frac{J_p}{m} + \frac{J_d}{m} \right) \ddot{\theta} + \rho^T n \ddot{s} + J_d \ddot{\psi} + \tau^T \rho \dot{s} \dot{\theta} + \tau^T \rho \dot{\theta} \dot{s} + \rho \times k \dot{s}^2 + g \cdot (0, 1) \cdot T' \rho = u/m$$
$$\rho^T n \ddot{\theta} + \ddot{s} - \tau^T \rho \dot{\theta}^2 + g \cdot (0, 1) \cdot T \tau = \frac{F_t}{m R_{ball}}$$
$$J_d \ddot{\theta} + J_d \ddot{\psi} = F_t$$

# Mechanical Constraints

No slipping: velocity of the contact point

$$\psi = -\frac{s}{R_{ball}}$$

Holonomic constraint!

Lagrange Equations of the 1-st kind

$$\begin{aligned} & \left( \|\rho\|^2 + \frac{J_p}{m} + \frac{J_d}{m} \right) \ddot{\theta} + \rho^T n \ddot{s} + J_d \ddot{\psi} + \\ & \tau^T \rho \dot{s} \dot{\theta} + \tau^T \rho \dot{\theta} \dot{s} + \rho \times k \dot{s}^2 + g \cdot (0, 1) \cdot T' \rho = u/m \\ & \rho^T n \ddot{\theta} + \ddot{s} - \tau^T \rho \dot{\theta}^2 + g \cdot (0, 1) \cdot T \tau = \frac{F_t}{m R_{ball}} \\ & J_d \ddot{\theta} + J_d \ddot{\psi} = F_t \end{aligned}$$

# Mechanical Constraints

No slipping: velocity of the contact point

$$\psi = -\frac{s}{R_{ball}}$$

Holonomic constraint!

Lagrange Equations of the 1-st kind

$$\begin{aligned} & \left( \|\rho\|^2 + \frac{J_p}{m} + \frac{J_d}{m} \right) \ddot{\theta} + \rho^T n \ddot{s} + J_d \ddot{\psi} + \\ & \tau^T \rho \dot{s} \dot{\theta} + \tau^T \rho \dot{\theta} \dot{s} + \rho \times k \dot{s}^2 + g \cdot (0, 1) \cdot T' \rho = u/m \\ & \rho^T n \ddot{\theta} + \ddot{s} - \tau^T \rho \dot{\theta}^2 + g \cdot (0, 1) \cdot T \tau = \frac{F_t}{m R_{ball}} \\ & J_d \ddot{\theta} + J_d \ddot{\psi} = F_t \end{aligned}$$

## Mechanical Constraints

Tangential component of the reaction force:

$$F_t = \frac{J_d m}{1 + mR} \left( (\rho^T n + R_{ball}) \ddot{\theta} - \tau^T \rho \dot{\theta}^2 + g \cdot (0, 1) \cdot T \tau \right)$$

or

$$F_t \approx \frac{J_d m}{1 + mR} \left( (\rho^T n + R_{ball}) \frac{u}{J_p} - \tau^T \rho \dot{\theta}^2 + g (0, 1) T \tau \right)$$

Rolling of a disk over a rotating curve

$$\left(\|\rho\|^2 + \frac{J_p}{m} + \frac{J_d}{m}\right) \ddot{\theta} + \left(\rho \times \tau - \frac{J_d}{mR}\right) \ddot{s} + 2\tau^T \rho \dot{s} \dot{\theta} + \rho \times k \dot{s}^2 + g \cdot (0, 1) \cdot T' \rho = u/m$$
$$\left(\rho \times \tau - \frac{J_d}{mR}\right) \ddot{\theta} + \left(1 + \frac{J_d}{mR^2}\right) \ddot{s} - \tau^T \rho \dot{\theta}^2 + g \cdot (0, 1) \cdot T \tau = 0$$

and reaction forces

$$F_t = \frac{J_d m}{1 + mR} \left( \left( \rho^T n + R_{ball} \right) \frac{u}{J_p} - \tau^T \rho \dot{\theta}^2 + g \cdot (0, 1) \cdot T \tau \right)$$
$$F_n = m \left( -\rho^T \tau \frac{u}{J_p} - \rho^T n \dot{\theta}^2 - \tau \times k \dot{s}^2 - 2\dot{s} \dot{\theta} + g \cdot (0, 1) \cdot T n \right)$$

it must be

$$F_n > 0$$

$$|F_t| < F_{friction}$$

**We are able to control the disk motions!**

# Motion Planning

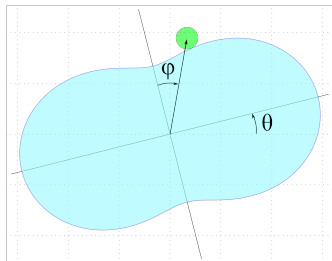
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## Generalized coordinates

We cannot measure the coordinate  $s$ !

Curve reparametrization  $s = s(\phi)$

Then  $q = (\theta, \phi)$



$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = (u, 0)^T$$

with

$$M(q) = m \begin{pmatrix} \|\rho\|^2 + \frac{J_{disk}}{m} + \frac{J_{ball}}{m} & \left(\rho \times \tau - \frac{J_{ball}}{mR}\right) s_\phi \\ \left(\rho \times \tau - \frac{J_{ball}}{mR}\right) s_\phi & \left(1 + \frac{J_{ball}}{mR^2}\right) s_\phi^2 \end{pmatrix}$$
$$C(q, \dot{q}) = m \begin{pmatrix} s_\phi \tau^T \rho \dot{\phi} & s_\phi \tau^T \rho \dot{\theta} + \left(s_\phi^2 \rho \times \kappa + \left(\rho \times \tau - \frac{J_{ball}}{mR}\right) s_{\phi\phi}\right) \dot{\phi} \\ -s_\phi \tau^T \rho \dot{\theta} & \left(1 + \frac{J_{ball}}{mR^2}\right) s_\phi s_{\phi\phi} \dot{\phi} \end{pmatrix}$$
$$G(q) = mg \begin{pmatrix} (0, 1)^T \rho \\ (0, 1)^T \tau s_\phi \end{pmatrix}$$

Let there is a relation

$$\theta = \Theta(\phi)$$

it gives a motion generator equation

$$\alpha(\phi)\ddot{\phi} + \beta(\phi)\dot{\phi}^2 + \gamma(\phi) = 0$$

with

$$\alpha = ms_{\phi} \left( \left( \rho \times \tau - \frac{J_{ball}}{mR} \right) \Theta' + \left( 1 + \frac{J_{ball}}{mR^2} \right) s_{\phi} \right)$$

$$\beta = ms_{\phi} \left( \left( \rho \times \tau - \frac{J_{ball}}{mR} \right) \Theta'' - \tau^T \rho \Theta'^2 + \left( 1 + \frac{J_{ball}}{mR^2} \right) s_{\phi\phi} \right)$$

$$\gamma = ms_{\phi} g \cdot (0, 1) \cdot T(\Theta)\tau$$

**how to choose  $\Theta$ ?**

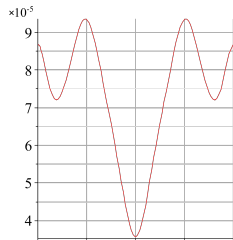
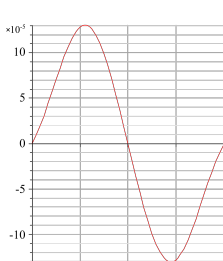
## Servo-Constraint

Picard-Lindelof theorem requires:  $\alpha, \beta, \gamma \in C^0(\mathbb{R})$  and  $\alpha(\mathbb{R}) \neq 0$   
then

$$\left(\rho \times \tau - \frac{J_{ball}}{mR}\right) \Theta' + \left(1 + \frac{J_{ball}}{mR^2}\right) s_\phi \neq 0$$

we could take  $\gamma = 0$  but  
this leads to  $\alpha = 0$  at  
some points!

Idea:  $\gamma =$  some function  
of  $\phi$



The nominal trajectory must preserve the constraints

How to take into account the constraints

$$F_t = \frac{J_d m}{1 + mR} \left( \left( \rho^T n + R_{ball} \right) \frac{u}{J_p} - \tau^T \rho \dot{\theta}^2 + g \cdot (0, 1) \cdot T \tau \right)$$

$$F_n = m \left( -\rho^T \tau \frac{u}{J_p} - \rho^T n \dot{\theta}^2 - \tau \times k \dot{s}^2 - 2\dot{s}\dot{\theta} + g \cdot (0, 1) \cdot T n \right)$$

$$|F_t| < F_{friction}$$

$$F_n > 0$$

?

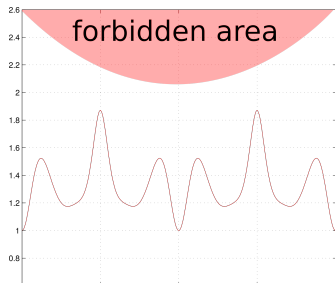
Substituting the servo-constraint into the inequalities we have We can find

$$\left| F_t(\phi, \dot{\phi}, \Theta, \Theta' \dot{\phi}) \right| < F_{friction}$$

and

$$F_n(\phi, \dot{\phi}, \Theta, \Theta' \dot{\phi}) > 0$$

they define a forbidden areas on the phase plane



# Motion Planning

