Lecture 9

Feedback Orbital Stabilization

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Learning outcomes: Ad-hoc approach for stabilizing of periodic motions of pendulum, A general approach, LQR for time-varying system, Matrix Riccati differential equation with periodic coefficients, Linear Hamiltonian equation

Outline

Feedback Orbital Stabilization

Motion Control: Top View

Euler-Lagrange systems

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q)u$$

1. T-periodic trajectory planning using servo-constraints: $q_\star(t), u_\star(t)$
2. Stability analysis
   2.1 Transverse coordinates:
   $$(q,\dot{q}) \mapsto (y,\dot{y},I)$$
   2.2 Transverse linearization:
   $$\dot{z} = A(t)z$$
   2.3 Andronov-Witt theorem
3. Synthesis of feedback controller?

Transverse Dynamics

$$\frac{dz}{dt} = A(t)z + B(t)v \tag{1}$$

with $z = (y, \dot{y}, I) \in \mathbb{R}^{2n-1}$, $v \in \mathbb{R}^m$

Let there is $v(t,z)$ s.t. solution $z = 0$ of (1) is stable

Algorithm:

1. Measure $q, \dot{q}$
2. Find the transformation: $(q, \dot{q}) \mapsto (t, z)$
3. Evaluate control signal of linear system: $v(t,z)$
4. Evaluate original control signal: $u = u(q, \dot{q}, v)$

What is t? How to find t?

Time projection

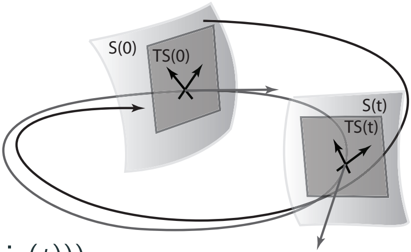
We know the transformation $(z, t) \mapsto (q, \dot{q})$:

$$(q, \dot{q}) = e_1(t)z_1 + e_2(t)z_2 + ... + e_{2n-1}(t)z_{2n-1}$$

where e_1, e_2 are the basis of Poincare hyperplane

- forward transformation $(z, t) \mapsto (q, \dot{q})$
- inverse transformation $(q, \dot{q}) \mapsto (z, t)$ exists if Jacobian is non-degenerate
- t can be found as

$$t = \arg \min_{t \in [0, T)} \operatorname{dist}((q, \dot{q}), (q_\star(t), \dot{q}_\star(t)))$$



Example: Pendulum Oscillations

Generalized coordinates: θ

External torque: u

Lagrangian: $L = \frac{\dot{\theta}^2}{2} + k\cos\theta$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \ddot{\theta}$$
$$\frac{\partial L}{\partial \theta} = -k\sin\theta$$

Equations of motion

$$\ddot{\theta} + k\sin\theta = u$$

or in a matrix form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$$

with $M = [1]\,, C = [0]\,, G = [k\sin\theta],\ q = \theta$

Orbital Stabilization of Pendulum Oscillations

$$\ddot{\theta} + k \sin\theta = u$$

Let us consider its particular solution:

$$u_\star(t) = 0$$
$$\theta_\star(t) \quad \text{some function}$$

We know that the system has the first integral, the total energy

$$E\left(\theta, \dot{\theta}\right) = \frac{\dot{\theta}^2}{2} - k\cos\theta$$

so we can find

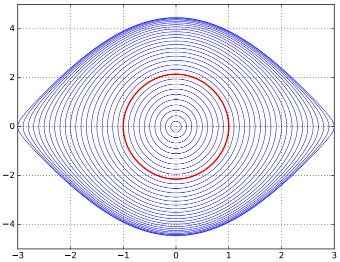
$$E_\star = \frac{\dot{\theta}_\star^2(t)}{2} - k\cos\theta_\star(t)$$

Geometrical Meaning

On the different trajectories the function $E\left(\theta, \dot{\theta}\right)$ will have the different values

$\Rightarrow$ the function uniquely defines in a small vicinity the necessary trajectory

this function can be thought as a perpendicular coordinate



is energy a transverse coordinate?

Geometrical Meaning

On the other hand on the previous lecture we introduced the coordinates

$$z = (y, \dot{y}, I)$$

For n-DOF system: dim of phase space $2n$, and $\dim z = 2n - 1$
For the pendulum $n = 1$, then $\dim z = 1$

There are no servo-constraints, so we don't have y, $\dot{y}$, and the only coordinate is I

$$\begin{aligned} I\left(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0\right) &= \dot{\theta}^2 - \left(\dot{\theta}_0^2 - 2\int_{\theta_0}^{\theta} \frac{\gamma(v)}{\alpha(v)} \psi(\theta_0, v)\, dv\right) \psi(\theta, \theta_0) \\ &= \dot{\theta}^2 - 2k\cos\theta - \dot{\theta}_0^2 + 2k\cos\theta_0 \\ &= 2E - 2E_0 \end{aligned}$$

with $\psi(\theta_0, v) = \exp\left\{2\int_{\theta_0}^{v} \frac{\beta(w)}{\alpha(w)} dw\right\}$

Transverse Coordinates

New coordinates for the pendulum: $\left(\theta, \dot{\theta}\right) \mapsto (\theta, I)$ Dynamics in new coordinates:

$$\frac{dI}{dt} = \frac{d}{dt}(2E - 2E_0) = 2\frac{dE}{dt}$$

and

$$\frac{dE}{dt} = \dot{\theta}\ddot{\theta} + k \sin\theta\dot{\theta} = \dot{\theta} \underbrace{\left(\ddot{\theta} + k \sin\theta\right)}_{\text{from the equations of motion}} = \dot{\theta} u$$

then

$$\frac{dI}{dt} = 2\dot{\theta} u$$

Transverse Dynamics

Transverse dynamics

$$\frac{dI}{dt} = 2\dot{\theta} u$$

and its linearization is

$$\frac{dI}{dt} = 2\dot{\theta}_{\star}(t)\, u$$

is like

$$\frac{dz}{dt} = A(t)z + B(t)u$$

so we have $A(t) = 0 \in \mathbb{R}^{1\times 1}$, $B(t) = 2\dot{\theta}_{\star}(t) \in \mathbb{R}^{1\times 1}$

Transverse Feedback Stabilization

The dynamics

$$\frac{dI}{dt} = 2\dot{\theta}_\star(t)\,u$$

can be stabilized easily by

$$u(t, I) = -c \cdot \dot{\theta}_\star(t) \cdot I$$

or

$$u\left(t, \theta, \dot{\theta}\right) = -c \cdot \dot{\theta}_\star(t) \cdot (2E - 2E_0)$$

This is exactly the energy-based control!

The energy plays the role of a transverse coordinate!

Transverse Feedback Stabilization

Is system

$$\frac{dI}{dt} = 2\dot{\theta}_\star(t)\, u$$
$$u(t, I) = -c \cdot \dot{\theta}_\star(t) \cdot I$$

Stable? The Lyapunov function candidate is

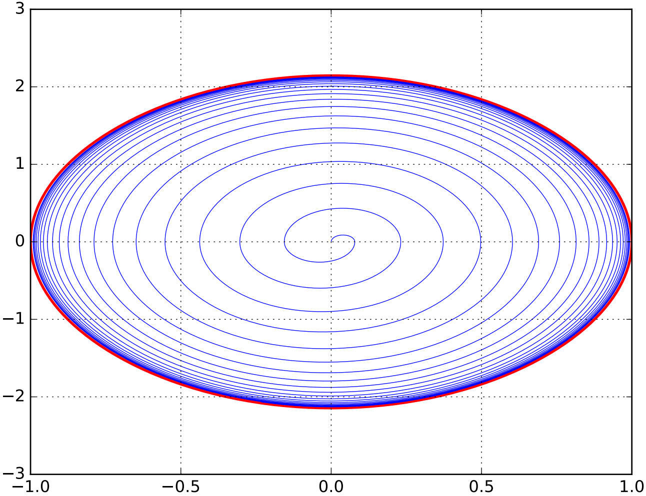
$$V = \frac{1}{2} I^2$$

then

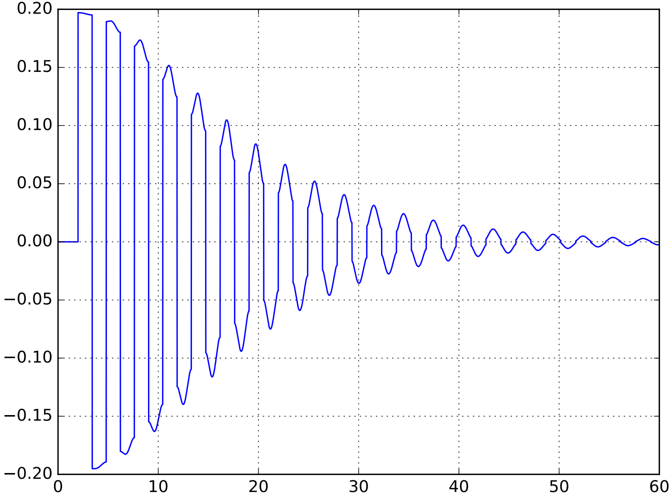
$$\dot{V} = I\dot{I} = -2c\dot{\theta}_\star^2(t)\, I^2$$

is negative definite for non-trivial solutions $\theta_\star(t)$

Transient Process



Control Input



Pendulum Animation

Transverse Feedback Stabilization

In general case:

$$\frac{dz}{dt} = A(t)z + B(t)u$$

$$\dim z = 2n - 1,$$
$$\dim A = 2n - 1 \times 2n - 1,$$
$$\dim B = 2n - 1 \times m$$

it's necessary to find control input u in a form

$$u = k(t) \cdot z$$

with $k : \mathbb{R} \to \mathbb{R}^{m \times n}$

$$\frac{dz}{dt} = A(t)\,z + B(t)\,k(t)\,z = [A(t) + B(t)\,k(t)]\,z$$

Transverse Feedback Stabilization

The same trick doesn't work: if we take

$$k(t) = -c \cdot B^T(t)$$

then

$$\dot{V} = z^T \dot{z} = z^T \underbrace{\left[A(t) - c \cdot B(t) B^T(t)\right]}_{\text{not necessary negative-definite!}} z$$

how to find k(t)?

Matrix Riccati Differential Equation

The Lyapunov Function Method

$$\frac{dz}{dt} = A(t)\,z + B(t)\,u$$
$$u = k(t)z$$

The typical approach of the control theory: the Lyapunov functions method.

We consider the Lyapunov function candidate of the form

$$V = z^T R(t)\,z$$

with some positive definite symmetric matrix function $R(t) \in \mathbb{R}^{2n-1 \times 2n-1}$

The Lyapunov Function Method

Time derivative of the Lyapunov function wrt linear system

$$\frac{dV}{dt} = \dot{z}^T Rz + z^T R\dot{z} + z^T \frac{dR}{dt} z = -z^T Qz$$

Q is a positive definite matrix[1].

Substitute $\dot{z}$ into the the expression:

$$z^T A^T Rz + u^T B^T Rz + z^T RAz + z^T RBu + z^T \dot{R}z + z^T Qz = 0$$

Let the control input is of the form

$$u = -\frac{1}{\Gamma} B^T Rz$$

with some $\Gamma > 0$

[1] for brevity the argument t omitted: $R = R(t), B = B(t), A = A(t), V = V(t)$

Differential Riccati Equation

$$z^T A^T R z - \frac{1}{\Gamma} z^T R^T B B^T R z + z^T R A z - \frac{1}{\Gamma} z^T R B B^T R z + z^T \dot{R} z + z^T Q z = 0$$

quadratic form equals 0 for any z:

$$A^T R - \frac{1}{\Gamma} R^T B B^T R + RA - \frac{1}{\Gamma} R B B^T R + \dot{R} + Q = 0$$

remember that $R^T = R$, collect the terms:

$$\dot{R} + A^T R + RA - R \underbrace{\left(\frac{2}{\Gamma} B B^T\right)}_{=: P(t)} R + Q = 0$$

then

$$\frac{dR}{dt} + A^T R + RA - RPR + Q = 0$$

Differential Riccati Equation

$$\frac{dR(t)}{dt} + A^T(t)\,R(t) + R(t)\,A(t) - R(t)\,P(t)\,R(t) + Q = 0$$

- differential equation
- first order: $R(0)$ uniquely defines solution
- nonlinear: there is a bilinear term $R \cdot P \cdot R$
- non-autonomous: coefficients depend on time
- all the coefficients are periodic functions

Matrix Riccati Differential Equation
with Periodic Coefficients

How to solve MRDEwPC?

Integrate numerically?
Take any $R(0)$, and find corresponding $R(t)$?

Example

MRDE with $Q = -1, A = 0, P = 1$:

$$\dot{R} = R^2 + 1$$

it's solution

$$R = \tan(t - t_0)$$

is unbounded $\Rightarrow$ doesn't fit the requirements

How to solve MRDEwPC?

Solution of the DE must be

- square matrix function of dimension $2n-1 \times 2n-1$
- positive-definite: Lyapunov function must be positive definite

$$V = z^T R(t) z$$

- periodic: cannot integrate in infinite time
- bounded: control input must be bounded

$$u = -\frac{1}{\Gamma} B^T(t) R(t) z$$

Solution of the MRDEwPC

Solution of the MRDEwPC

How to transform

$$\dot{R} + A^T R + RA - RPR + Q = 0$$

to a linear DE? Let

$$R = NU^{-1}$$

for some matrices U, N of the same dimension[2]

If $U(t), N(t)$ found, then it's possible to reconstruct $R(t)$!

Substitute

$$\frac{dR}{dt} = \frac{dN}{dt} U^{-1} + N \frac{dU^{-1}}{dt}$$

$$\frac{dN}{dt} U^{-1} + N \frac{dU^{-1}}{dt} + A^T N U^{-1} + N U^{-1} A - N U^{-1} P N U^{-1} + Q = 0$$

[2]keep in mind: $U = U(t), N = N(t)$

Solution of the MRDEwPC

By multiply by U on the right, have:

$$\frac{dN}{dt} + N\frac{dU^{-1}}{dt}U + A^T N + NU^{-1}AU - NU^{-1}PN + QU = 0$$

taking into account that

$$\frac{dU^{-1}U}{dt} = \frac{dU^{-1}}{dt}U + U^{-1}\frac{dU}{dt} = 0 \Rightarrow \frac{dU^{-1}}{dt} = -U^{-1}\frac{dU}{dt}U^{-1}$$

we have

$$\frac{dN}{dt} - NU^{-1}\frac{dU}{dt} + A^T N + NU^{-1}AU - NU^{-1}PN + QU = 0$$

we have excessive variables! Additional constraints

$$\frac{dN}{dt} + A^T N + QU + \underbrace{NU^{-1}\left[-\frac{dU}{dt} + AU - PN\right]}_{\text{require this is 0}} = 0$$

Hamiltonian Equation

$$\frac{dU}{dt} = AU - PN$$
$$\frac{dN}{dt} = -A^T N - QU$$

or in matrix form

$$\underbrace{\frac{d}{dt}\begin{bmatrix} U \\ N \end{bmatrix}}_{=:X} = \underbrace{\begin{bmatrix} A & -P \\ -Q & -A^T \end{bmatrix}}_{=:H} \begin{bmatrix} U \\ N \end{bmatrix}$$

Conclusion: we transformed MRDE into a linear DE of 2×-more dimension

$$\dot{X} = H(t)X$$

Hamiltonian Equation

$$\dot{R} + A^T R + RA - RPR + Q = 0$$

transformed into

$$\dot{X} = H(t)X \quad \text{with} \quad X = \begin{bmatrix} U \\ N \end{bmatrix}$$

- linear equation: any solution is a linear combination of fundamental $X(t) = \Phi(t)X(0)$
- periodic coefficients: $H(t+T) = H(t)$
- H is Hamiltonian matrix: $(JH)^T = JH$ with $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$
- the fundamental solution $\Phi(t)$ is symplectic matrix: $\Phi^T J \Phi = J$
- half of the solutions stable, and half are unstable

Stable Solutions of Hamiltonian Equation

1. find the monodromy matrix $\Phi(T)$ by integrating $\dot{\Phi} = H(t)\Phi$ with $\Phi(0) = I$
2. $\Phi(t)$ has $2n$ eigenvalues: $\lambda_1, \frac{1}{\lambda_1}, \lambda_2, \frac{1}{\lambda_2}, \dots$
3. take eigenvectors v_i corresponding to: $|\lambda_i| \leq 1$
4. find the basis of the stable subspace: $\mathrm{span}(v_1, v_2, \dots, v_n)$
5. the stable solutions of Hamiltonian equation are given by the i.c. $X(0) = \mathrm{span}(v_1, v_2, \dots, v_n)$
6. the corresponding bounded solutions of the MRDEwPC are given by the i.c.

$$\begin{bmatrix} U_0 \\ N_0 \end{bmatrix} = X_0 = \mathrm{span}(v_1, v_2, \dots, v_n)$$

The Butterfly Robot Transverse Dynamics

The Butterfly Robot Transverse Coordinates

There are 2 trajectories satisfying

$$I(\phi, \dot{\phi}, \phi_0, \dot{\phi}_0) = 0 : \quad \phi_\star(t) \text{ and } -\phi_\star(t)$$

Moreover, transverse dynamics for both of them are the same

Consequence of local stability

We need rotation to a one side!

How to change transverse coordinates?

New transverse coordinates

$$I(\phi, \dot{\phi}, \phi_0, \dot{\phi}_0) = \dot{\phi}^2(\phi) - \dot{\phi}_\star^2(\phi, \phi_0, \dot{\phi}_0)$$

What if take a coordinate

$$z = \dot{\phi}\left(\phi\right) - \dot{\phi}_\star\left(\phi, \phi_0, \dot{\phi}_0\right)?$$

$$\frac{dz}{dt} = -\frac{\beta\left(\phi\right)\dot{\phi}^2 + \gamma\left(\phi\right) + v}{\alpha\left(\phi\right)} - \frac{d\dot{\phi}_\star\left(\phi, \phi_0, \dot{\phi}_0\right)}{d\phi}\dot{\phi}$$

used

$$\alpha(\phi)\ddot{\phi} + \beta(\phi)\dot{\phi}^2 + \gamma(\phi) = v$$

Linearized transverse dynamics

Linearized transverse dynamics:

$$\frac{dz}{dt} = A(t)z + B(t)u$$

$$A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \frac{g_y(\phi_\star(t))}{\alpha(\phi_\star(t))} & \frac{g_{\dot{y}}(\phi_\star(t))}{\alpha(\phi_\star(t))} & \frac{\gamma(\phi_\star(t)) - \beta(\phi_\star(t))\dot{\varphi}_\star^2}{\alpha(\phi_\star(t))\dot{\varphi}_\star} \end{pmatrix}$$

$$b(t) = \begin{pmatrix} 0 \\ 1 \\ \frac{g_v(\phi_\star(t))}{\alpha(\phi_\star(t))} \end{pmatrix}$$

Feedback controller coefficients and components of the Riccati Matrix