Lecture 2

Analytic Tools for Analysis of the 2nd Order Systems

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UNAM, Mexico
November 26, 2018

Learning outcomes: Concepts of stability of a motion. Lyapunov's conditions for revealing a center by linearization. A Poincare first return map for analysis of stability of a periodic solution and its linearization. Examples.

Outline

1. Orbital Stability vs. Stability of a Motion

1.1 Definitions

1.2 Preliminary comments

2. Lyapunov's theorems

3. Poincare's arguments on stability

Orbital Stability vs. Stability of a Motion

Orbital Stability vs. Stability of a Motion

Consider a nonlinear dynamic system and one of its solutions

$$\tfrac{d}{dt}x = f(x), \quad x^*(t) = x^*(t, x_0^*) \in \mathbb{R}^n,\ \ t \in [0, +\infty)$$

Definition (Lyapunov Stability)

The solution $x^*(\cdot)$ is called

- stable, if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that,

$$\text{if} \quad \|x(0) - x_0^*\| < \delta, \quad \text{then} \quad \|x(t) - x^*(t)\| < \varepsilon \quad \forall\, t \geq 0$$

- asymptotically stable, if, in addition,

$$\lim_{t \to +\infty} \|x(t) - x^*(t)\| = 0.$$

Orbital Stability vs. Stability of a Motion

Consider a nonlinear system and one of its periodic solutions

$$\frac{d}{dt}x = f(x), \quad x^*(t) = x^*(t, x_0^*), \quad x^*(t) = x^*(t+T), \ \forall\, t, \ \ T > 0$$

Let $\Gamma_{x^*} \subset \mathbb{R}^n$ denote the orbit of $x^*(\cdot)$

$$\Gamma_{x^*} = \left\{ \xi \in \mathbb{R}^n : \ \xi = x^*(t), \ t \in [0, T] \right\}.$$

Definition (Orbital Stability)

The periodic solution $x^*(\cdot)$ is called

- orbitally stable, if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that,

$$\text{if} \quad \|x(0) - x_0^*\| < \delta, \quad \text{then} \quad \operatorname{dist}\left(x(t), \Gamma_{x^*}\right) < \varepsilon \quad \forall\, t \geq 0$$

- asymptotically orbitally stable, if, in addition,

$$\lim_{t \to +\infty} \operatorname{dist}\left(x(t), \Gamma_{x^*}\right) = 0$$

Important observations

The method developed in the course for planning behaviors of under-actuated mechanical systems with many degrees of freedom will be based

on analysis of low dimensional subsystems

that allow reproducing individual motions.

Important observations

The hypothesis of reduction of dimension in generating behaviors of complex systems is not new.

But the lectures will advocate for specific arguments in realizing such reduction, where the corresponding low dimensional system is

always the scalar 2nd order system.

Important observations

Arguments developed for motion planing should be complemented

- by controller design steps and
- by analysis of the closed loop system stability.

We will bring your attention to classical settings developed by A. Poincare in analyzing

orbital stability of periodic behaviors

again for scalar 2nd order systems.

Searching for an Equilibrium vs. Detecting a Cycle

To find an equilibrium x_0 of the dynamical system

$$\frac{d}{dt}x = f(x)$$

one needs only to solve algebraic equation

$$f(x_0) = 0$$

To find a T-periodic solution $x(t) = x(t+T)$ of the system

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one needs, in general, to integrate analytically this system

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Except particular cases, it is impossible!

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Lyapunov's theorems

Lyapunov theorems. Preliminaries:

Consider a 2nd order system

$$\begin{aligned}\dot{x} &= P(x, y)\\ \dot{y} &= Q(x, y)\end{aligned}$$

Suppose that (x_0, y_0) is an equilibrium, i.e. the point where

$$P(x_0, y_0) = Q(x_0, y_0) = 0$$

Introduce the linearization of the system at (x_0, y_0)

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P'_x(x_0, y_0) & P'_y(x_0, y_0) \\ Q'_x(x_0, y_0) & Q'_y(x_0, y_0) \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

If

$$P'_x(x_0, y_0)Q'_y(x_0, y_0) - P'_y(x_0, y_0)Q'_x(x_0, y_0) \neq 0$$

the equilibrium (x_0, y_0) is called simple.

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Lyapunov theorems. Preliminaries (Cont'd):

Consider the case when the linearized system is

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad b > 0$$

It is easy to check that:

All solutions are periodic

All solutions are of the same period $T = \frac{2\pi}{b}$

The function

$$z_1^2(t) + z_2^2(t) = C = z_1^2(0) + z_2^2(0)$$

is the integral of the system

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The linearization is a cheap computational procedure

Question: Is it true that if the linearization

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has a periodic solution, then the original nonlinear system

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned}$$

has one too?

Answer: Unfortunately, NO!

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Example:

Consider the system

$$\begin{aligned}\dot{x} &= -y - x(x^2 + y^2)\\ \dot{y} &= x - y(x^2 + y^2)\end{aligned}$$

Its linearization at the equilibrium $x = y = 0$

$$\begin{aligned}\dot{z}_1 &= -z_2\\ \dot{z}_2 &= z_1\end{aligned}$$

has the centre at $z_1 = z_2 = 0$!

Unfortunately, this nonlinear system has

NO periodic motion at all !!!

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Example (Cont'd):

Indeed, in the polar coordinates

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

the differential equation for radius ρ is decoupled

$$\begin{cases} \dot{x} = -y - x(x^2 + y^2) \\ \dot{y} = x - y(x^2 + y^2) \end{cases} \quad \Rightarrow \quad \dot{\rho} = -\rho^3$$

The solution of the last equation is

$$\rho(t) = \frac{\rho(0)}{\sqrt{2t\cdot\rho^2(0) + 1}} \to 0 \quad \text{as} \quad t \to +\infty$$

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Conclusions from the Example:

- Existence of a periodic solution for the linearized system

  DOES NOT IMPLY

  the presence of a cycle in the nonlinear original system

- Establishing a presence of a periodic solution of a nonlinear system by linearization is

  NONTRIVIAL TASK

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Lyapunov Theorems. Preliminaries (Cont'd):

Consider the system

$$\begin{aligned} \dot{x} &= -by + \phi(x,y) \\ \dot{y} &= bx + \psi(x,y) \end{aligned}$$

Here $b > 0$ and the functions ϕ, ψ are analytical

$$\begin{aligned} \phi(x,y) &= P_2(x,y) + P_3(x,y) + \cdots + P_k(x,y) + \ldots \\ \psi(x,y) &= Q_2(x,y) + Q_3(x,y) + \cdots + Q_k(x,y) + \ldots \end{aligned}$$

where $P_k(x,y)$, $Q_k(x,y)$ are homogeneous polynomials of degree k

$$\begin{aligned} P_2(x,y) &= p_{21}\,x^2 + p_{22}\,xy + p_{23}\,y^2 \\ P_3(x,y) &= p_{31}\,x^3 + p_{32}\,x^2y + p_{33}\,xy^2 + p_{34}\,y^3 \\ &\vdots \end{aligned}$$

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Lyapunov Theorems. Preliminaries (Cont'd):

In the polar coordinates

$$x = \rho\cos\theta, \quad y = \rho\sin\theta$$

the dynamics

$$\begin{cases} \dot{x} &= -by + \phi(x,y) \\ \dot{y} &= bx + \psi(x,y) \end{cases}$$

is

$$\begin{cases} \dot{x} &= \dot{\rho}\cos\theta - \rho\sin\theta\dot{\theta} &= -b\rho\sin\theta + \phi(\rho\cos\theta,\rho\sin\theta) \\ \dot{y} &= \dot{\rho}\sin\theta + \rho\cos\theta\dot{\theta} &= b\rho\cos\theta + \psi(\rho\cos\theta,\rho\sin\theta) \end{cases}$$

Therefore

$$\begin{cases} \dot{\rho} = \dot{x}\cos\theta + \dot{y}\sin\theta = \phi(\rho\cos\theta,\rho\sin\theta)\cos\theta + \psi(\rho\cos\theta,\rho\sin\theta)\sin\theta \\ \dot{\theta} = \dfrac{\dot{y}\cos\theta - \dot{x}\sin\theta}{\rho} = b + \dfrac{\psi(\rho\cos\theta,\rho\sin\theta)\cos\theta - \phi(\rho\cos\theta,\rho\sin\theta)\sin\theta}{\rho} \end{cases}$$

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The equations

$$\begin{aligned}\dot\rho &= \phi(\rho\cos\theta,\rho\sin\theta)\cos\theta+\psi(\rho\cos\theta,\rho\sin\theta)\sin\theta\\ \dot\theta &= b+\frac{\psi(\rho\cos\theta,\rho\sin\theta)\cos\theta-\phi(\rho\cos\theta,\rho\sin\theta)\sin\theta}{\rho}\end{aligned}$$

can be rewritten as

$$\begin{aligned}\dot\rho &= \underbrace{\Big[P_2(x,y)+P_3(x,y)+\dots\Big]}_{\phi(x,y)}\cos\theta+\underbrace{\Big[Q_2(x,y)+Q_3(x,y)+\dots\Big]}_{\psi(x,y)}\sin\theta\\ &= \Big[P_2(\rho\cos\theta,\rho\sin\theta)+\dots\Big]\cos\theta+\Big[Q_2(\rho\cos\theta,\rho\sin\theta)+\dots\Big]\sin\theta\\ &= \rho^2\Big[P_2(\cos\theta,\sin\theta)\cos\theta+Q_2(\cos\theta,\sin\theta)\sin\theta\Big]+\dots\\ \dot\theta &= b+\rho\Big[Q_2(\cos\theta,\sin\theta)\cos\theta-P_2(\cos\theta,\sin\theta)\sin\theta\Big]+\dots\end{aligned}$$

Equilibrium $\rho=0$ is simple (i.e. $b>0$) $\Rightarrow$ $\dot\theta\neq 0$ for small ρ

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$$\begin{aligned}\dot{\rho} &= \phi(\rho\cos\theta, \rho\sin\theta)\cos\theta + \psi(\rho\cos\theta, \rho\sin\theta)\sin\theta \\ \dot{\theta} &= b + \frac{\psi(\rho\cos\theta, \rho\sin\theta)\cos\theta - \phi(\rho\cos\theta, \rho\sin\theta)\sin\theta}{\rho}\end{aligned}$$

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&= \Big[P_2(\rho\cos\theta,\rho\sin\theta)+\dots\Big]\cos\theta+\Big[Q_2(\rho\cos\theta,\rho\sin\theta)+\dots\Big]\sin\theta \\
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Let us exclude time t from consideration

$$\frac{d\rho}{d\theta} = \frac{\rho^2 \Big[P_2(\cos\theta, \sin\theta)\cos\theta + Q_2(\cos\theta, \sin\theta)\sin\theta\Big] + \ldots}{b + \rho\Big[Q_2(\cos\theta, \sin\theta)\cos\theta - P_2(\cos\theta, \sin\theta)\sin\theta\Big] + \ldots} = R(\rho, \theta)$$

The function in the right-hand side is

- periodic in θ: $R(\rho, \theta) = R(\rho, 2\pi + \theta)$
- for any θ: $R(\rho, \theta)\Big|_{\rho=0} = 0$
- analytic function in some vicinity of $\rho = 0$

$$R(\rho, \theta) = \rho R_1(\theta) + \rho^2 R_2(\theta) + \rho^3 R_3(\theta) + \ldots$$

Lyapunov Theorems. Preliminaries (Cont'd):

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$$R(\rho,\theta)=\rho R_1(\theta)+\rho^2 R_2(\theta)+\rho^3 R_3(\theta)+\dots$$

Lyapunov Theorems. Preliminaries (Cont'd):

Let

$$\rho(\theta) = f\left(\theta; \theta_0, \rho_0\right)$$

be a solution of the differential equation

$$\frac{d}{d\theta}\rho = R(\rho, \theta)$$

originated at ρ_0 when $\theta = \theta_0$. Then the following statements hold

- $f\left(\theta; \theta_0, \rho_0\right)\Big|_{\rho_0=0} = 0;$
- f is the analytic function

$$f\left(\theta; \theta_0, \rho_0\right) = u_1(\theta)\rho_0 + u_2(\theta)\rho_0^2 + \cdots + u_k(\theta)\rho_0^k + \dots$$

- all the weighting functions in the expansion

$$u_1(\theta), \quad u_2(\theta), \quad \dots, \quad u_k(\theta), \quad \dots$$

can be recursively found.

Lyapunov Theorems. Preliminaries (Cont'd):

Let

$$\rho(\theta) = f\left(\theta; \theta_0, \rho_0\right)$$

be a solution of the differential equation

$$\tfrac{d}{d\theta}\rho = R(\rho, \theta)$$

originated at ρ_0 when $\theta = \theta_0$. Then the following statements hold

- $f\left(\theta; \theta_0, \rho_0\right)\Big|_{\rho_0=0} = 0$;
- f is the analytic function

$$f\left(\theta; \theta_0, \rho_0\right) = u_1(\theta)\rho_0 + u_2(\theta)\rho_0^2 + \cdots + u_k(\theta)\rho_0^k + \ldots$$

- all the weighting functions in the expansion

$$u_1(\theta), \quad u_2(\theta), \quad \ldots, \quad u_k(\theta), \quad \ldots$$

can be recursively found.

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First Lyapunov Theorem

The function

$$f\left(\theta; \theta_0, \rho_0\right)\Big|_{\theta=2\pi,\,\theta_0=0} = u_1(2\pi)\rho_0 + u_2(2\pi)\rho_0^2 + \dots$$
$$= \alpha_1\rho_0 + \alpha_2\rho_0^2 + \dots + \alpha_k\rho_0^k + \dots$$

is referred to as the Poincare first return map; the constants $\{\alpha_1, \alpha_2, \dots\}$ are known as focus quantities.

Theorem: *The first nonzero coefficient of the series*

$$f\left(2\pi; 0, \rho_0\right) - \rho_0 = (\alpha_1 - 1)\rho_0 + \alpha_2\rho_0^2 + \dots + \alpha_k\rho_0^k + \dots$$

has an odd number. ■

Q: When does the system have the center at its equilibrium?

A: If and only if $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \dots = 0$

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The necessary and sufficient conditions for the system

$$\begin{aligned} \dot{x} &= -by + \phi(x, y) \\ \dot{y} &= bx + \psi(x, y) \end{aligned}$$

to have the center at its equilibrium (x_0, y_0) *provided that* $\phi(x, y)$, $\psi(x, y)$ *are analytic functions, are that in some neighborhood of* (x_0, y_0) *the system has an integral* $H(x, y)$, *which is an analytic function. Furthermore, it has the form*

$$x^2 + y^2 + \Phi_3(x, y) + \Phi_4(x, y) + \cdots + \Phi_k(x, y) + \cdots = C$$

where $\Phi_k(x, y)$ *are homogeneous polynomials of degree* k. ■

Important: If the functions $\phi(x, y)$, $\psi(x, y)$ are not analytic , then the statement is not true!

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Example:

The dynamics of a point-mass pendulum is

$$\ddot{\theta} + a \cdot \sin\theta = 0$$

Its linearization around the downward equilibrium $\theta = 0$ is

$$\ddot{z} + a \cdot z = 0$$

The pendulum has its energy $E(\cdot)$ as an integral of the motion and

$$2 \cdot E(\theta, \dot{\theta}) = \dot{\theta}^2 + 2a\,(1 - \cos\theta) = \dot{\theta}^2 + a \cdot \theta^2 + \ldots$$

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Poincare's arguments on stability

Poincare First Return Map

Consider the system
$$\dot{x} = P(x,y)$$
$$\dot{y} = Q(x,y)$$

Suppose that there is a line l transverse to its solutions.

Introduce a scalar parameter s that uniquely defined points on l.

Suppose a trajectory L

- intersects l in the point s_0 when $t = t_0$
- intersects l one more time in the point s_1 when $t = t_1 > t_0$

$\Rightarrow$ There is a vicinity of s_0 such that solutions originated from that subset of l intersect l one more time, i.e. the map $f : l \to l$

$$s \mapsto \bar{s} = f(s), \quad \forall\, s \in (s_0 - \varepsilon_1,\, s_0 + \varepsilon_2)\,, \quad \varepsilon_1,\, \varepsilon_2 > 0$$

is well defined. It is the Poincare first return map.

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Poincare First Return Map (Cont'd)

Suppose that the trajectory L intersects the line l in the points

$$s_0,\ s_1,\ s_2,\ \ldots, s_k,\ \ldots$$

where $s_1 = f(s_0),\ s_2 = f(s_1),\ \ldots,\ s_k = f(s_{k-1}),\ \ldots$

Trajectory L converges to a cycle L^* $\Leftrightarrow$ $\lim_{k\to\infty} s_k = s^*,\ f(s^*) = s^*$

Definition:

The stationary point s^* of the Poincare first return map $f(\cdot)$, $s^* = f(s^*)$, is asymptotically stable, if there is its neighborhood $\mathcal{O} \subset l$, $s^* \in \mathcal{O}$, such that for any $s_0 \in \mathcal{O}$ the corresponding sequence $\{s_1, s_2, \ldots\}$ generated by $f(\cdot)$ converges to s^*

$$s_k \to s^* \quad as \quad k \to +\infty$$

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Stability of Poincare First Return Map

Lemma: *A stationary point s^* of the Poincare first return map*

$$s_{k+1} = f(s_k)$$

is stable, if

$$f'(s^*) < 1,$$

and unstable, if

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However, if

$$f'(s^*) = 1,$$

then the cycle is called complex, and one needs to consider higher derivatives of the map $f(\cdot)$ at $s = s^*$. ■

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Poincare First Return Map (Cont'd)

Given a system

$$\begin{aligned}\dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y),\end{aligned}$$

suppose the system has a T-periodic solution and it is known

$$x^* = \phi(t), \quad y^* = \psi(t), \quad t \in [0, T]$$

Q: How to compute the Poincare first return map $f(\cdot)$ for this periodic solution?

Q: How to check conditions of Lemma, i.e. how to compute $f'(\cdot)$?

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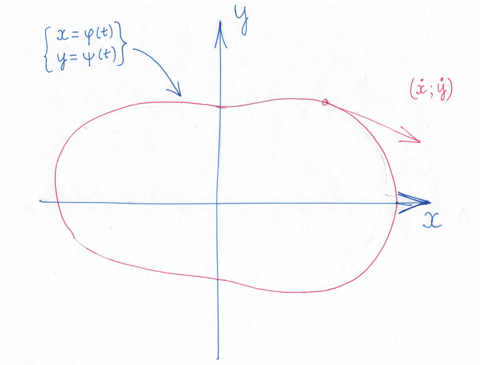
Coordinates in a Vicinity of a Cycle

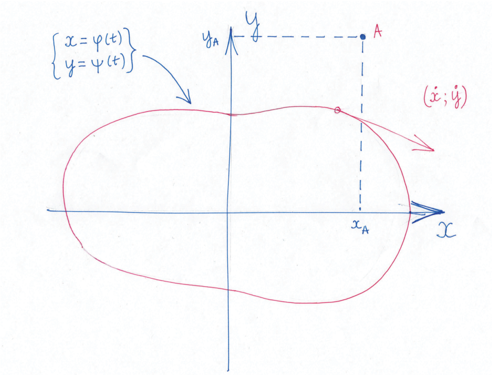


Given a cycle

$$x^* = \phi(t), \quad y^* = \psi(t),$$

we search for new (local) coordinates in its tubular vicinity.

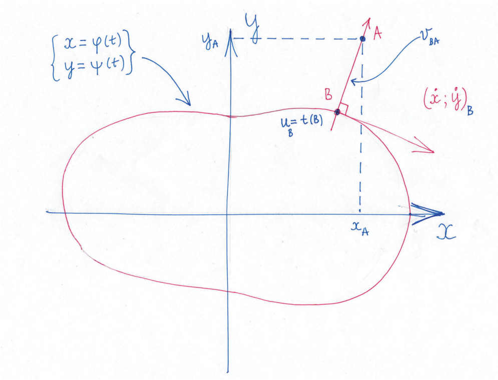
Coordinates in a Vicinity of a Cycle



Given a point A with coordinates $[x_A; y_A]$, we are to introduce two variables $[u_A; v_A]$, where

- v_A measures (somehow introduced) distance from A to the cycle
- u_A identifies (somehow introduced) projection point onto the cycle

Coordinates in a Vicinity of a Cycle



Given the point A, let B be the shortest point on the cycle to A.

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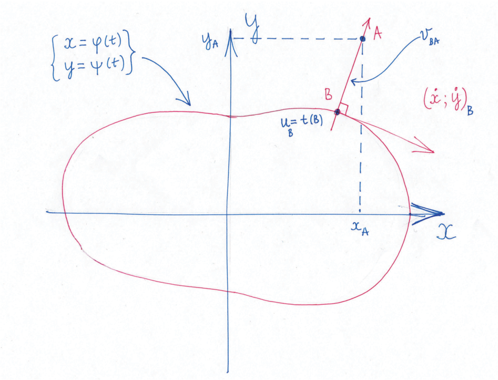

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The coordinates of B are $x_B = \phi(u_B)$, $y_B = \psi(u_B)$ or $u = u_B$, $v_B = 0$

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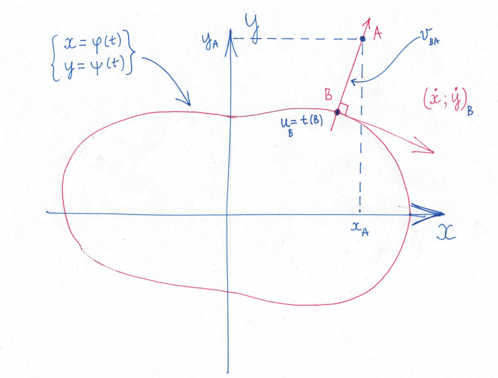

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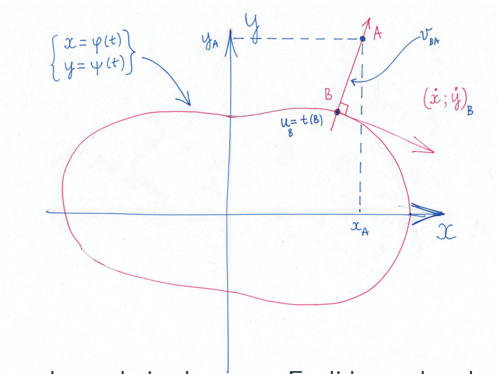
What are the coordinates $[u_A; v_A]$ if the distance is chosen as Euclidean?

Coordinates in a Vicinity of a Cycle



If the distance to the cycle is chosen as Euclidean, then locally

- $u_A = u_B$ and
- v_B equals to the size of the interval v_{BA} measured along the normal to the cycle at the point B

Poincare First Return Map (Cont'd)

To sum: new coordinates (u, v) in some vicinity of the cycle L^* with the following coordinates lines:

- $v(x, y) = C$ are closed curves around L^*;
- $v(x, y) = 0$ coincides with L^*;
- $u(x, y) = C$ are orthogonal to L^*.

One can think of (u, v) as polar coordinates (θ, ρ).

If a T-periodic solution is known

$$x^* = \phi(t), \quad y^* = \psi(t),$$

then a transformation into the new coordinates is defined by

$$\begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{bmatrix} \phi(u_A) \\ \psi(u_A) \end{bmatrix} + v_A \cdot \begin{bmatrix} -\psi'(u_A) \\ \phi'(u_A) \end{bmatrix}$$

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On the T-periodic solution

$$[x^* = \phi(t),\, y^* = \psi(t)] \quad \text{or the same} \quad [u^* = t,\, v^* \equiv 0]$$

the determinant of $J(\cdot)$ is

$$\det\ J(\cdot)\Big|_{u=u^*, v=v^*} = \phi'(t)^2 + \psi'(t)^2 > 0$$

Hence, the transform is well defined in its vicinity.

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Poincare First Return Map (Cont'd)

The system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

in new coordinates $[u, v]$ takes the form

$$\begin{aligned} \dot{u} &= U(u, v) \\ \dot{v} &= V(x, y) \end{aligned}$$

Furthermore, for analytic functions $P(\cdot)$, $Q(\cdot)$ the system can be rewritten as a scalar diff. equation

$$\frac{dv}{du} = \frac{V_1(u, v) \cdot v}{1 + U_1(u, v) \cdot v} = \underbrace{A_1(u)}_{= V_1(u,0)} \cdot v + A_2(u)v^2 + A_3(u)v^3 + \ldots$$

Poincare First Return Map (Cont'd)

The system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

in new coordinates $[u, v]$ takes the form

$$\begin{aligned} \dot{u} &= U(u, v) &= 1 + U_1(u, v) \cdot v \\ \dot{v} &= V(x, y) &= V_1(u, v) \cdot v \end{aligned}$$

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with initial conditions in $u_0 = 0$, v_0. It can be written as a series

$$v = f(u; 0, v_0) = a_1(u)v_0 + a_2(u)v_0^2 + \dots + a_k(u)v_0^k + \dots$$

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$$\frac{d}{du}\left[a_1(u)v_0 + a_2(u)v_0^2 + \dots + a_k(u)v_0^k + \dots\right] =$$

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This gives recurrent equations on unknown functions $a_i(u)$:

$$\begin{aligned} a_1'(u) &= A_1(u)a_1(u), \\ a_2'(u) &= A_1(u)a_2(u) + A_2(u)\left[a_1(u)\right]^2, \\ &\vdots \end{aligned}$$

where the initial conditions $a_1(0)$, $a_2(0)$, ... are defined by the series

$$v_0 = f(u; 0, v_0)\Big|_{u=0} = a_1(0)v_0 + a_2(0)v_0^2 + \cdots + a_k(0)v_0^k + \ldots$$

$$\Downarrow$$

$$a_1(0) = 1, \quad a_2(0) = a_3(0) = \cdots = a_k(0) = \cdots = 0$$

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Poincare First Return Map (Cont'd)

The Poincare first return map, defined on $I = \{u = 0\}$, will be

$$\begin{aligned} v = f(T; 0, v_0) &= a_1(T)v_0 + a_2(T)v_0^2 + \dots + a_k(T)v_0^k + \dots \\ &= \alpha_1 v_0 + \alpha_2 v_0^2 + \dots + \alpha_k v_0^k + \dots \end{aligned}$$

To check stability of a simple cycle we need to know only

$$\left. \frac{d\, f(T; 0, v_0)}{dv_0} \right|_{v_0 = 0} = \alpha_1$$

If $\alpha_1 < 1 \quad \Rightarrow \quad$ the cycle is stable

If $\alpha_1 > 1 \quad \Rightarrow \quad$ the cycle is unstable

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Poincare First Return Map (Cont'd)

The constant

$$\alpha_1 = a_1(T)$$

is known as a multiplier of the cycle.

To find it explicitly one can integrate the equation:

$$a_1' = A_1(u)a_1, \quad a_1(0) = 1.$$

The answer is

$$a_1(T) = \exp\left[\int_0^T A_1(u)du\right] \cdot 1$$

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Lemma:

$$\int_0^T A_1(u)du = \boxed{\int_0^T V_1(u,0)du = \int_0^T \left\{ P'_x(\phi(u),\psi(u)) + Q'_y(\phi(u),\psi(u)) \right\} du}$$

Hint: Linearizations of the system written in different coordinates

$$\dot{x} = P(x,y),\ \dot{y} = Q(x,y) \quad \text{or} \quad \dot{u} = U(u,v),\ \dot{v} = V(u,v)$$

have the same traces on the periodic solution

$$\mathrm{tr}\begin{bmatrix} \frac{\partial}{\partial x}P & \frac{\partial}{\partial y}P \\ \frac{\partial}{\partial x}Q & \frac{\partial}{\partial y}Q \end{bmatrix} = P'_x + Q'_y, \quad \mathrm{tr}\begin{bmatrix} \frac{\partial}{\partial u}U & \frac{\partial}{\partial v}U \\ \frac{\partial}{\partial u}V & \frac{\partial}{\partial v}V \end{bmatrix} = \frac{\partial}{\partial u}\left[U_1 \cdot v\right] + V_1$$

Therefore,

$$P'_x(x,y) + Q'_y(x,y)\Big|_{x=x^*(t),y=y^*(t)} \equiv V_1(u,v)\Big|_{u=u^*(t),v=v^*(t)=0}$$

Example:

$$\dot{x} = -y - \varepsilon x \left(x^2 + y^2 - 1\right) = P(x, y)$$
$$\dot{y} = x - \delta y \left(x^2 + y^2 - 1\right) = Q(x, y)$$

The system has a cycle

$$x = \phi(u) = \cos u, \quad y = \psi(u) = \sin u, \quad 0 \le u \le 2\pi$$

To verify (in)stability of this cycle, one can compute

$$\alpha_1 = \exp\left[\int_0^{2\pi} \left\{ P'_x(\phi(u), \psi(u)) + Q'_y(\phi(u), \psi(u)) \right\} du\right]$$
$$= \exp\left[\int_0^{2\pi} \left\{ -2\varepsilon \cos^2(u) - 2\delta \sin^2(u) \right\} du\right] = \exp\left[-2\pi(\varepsilon + \delta)\right]$$

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