Lecture 3

Constrained Mechanical Systems

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UNAM, Mexico
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Learning outcomes: Nonlinear mechanical systems with constraints. Classification of constraints. Stability of nonlinear mechanical systems with constraints. Examples.

Outline

Systems subject to holonomic constraints

Example: point-mass dynamics in excessive coordinates

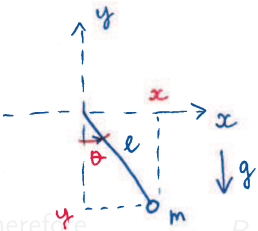
Let a point mass with coordinates (x, y) move in the vertical plane affected by the gravity. Assume that the distance of the point to the origin is kept constant

$$f\left(x(t), y(t)\right) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall\, t.$$

The dynamics of the point are

$$m \cdot \ddot{x} = R_x, \quad m \cdot \ddot{y} = R_y - m \cdot g,$$

where $R = [R_x;\ R_y]$ is the reaction force due to the constraint.



Therefore $R_x = \lambda \cdot x, \quad R_y = \lambda \cdot y.$

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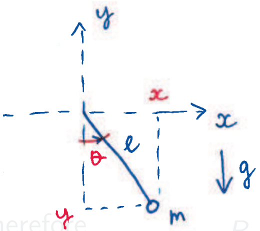

How to find the reaction force
$R = R(x, y, \dot{x}, \dot{y})$?

Therefore $\quad R_x = \lambda \cdot x, \quad R_y = \lambda \cdot y.$

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The constraint $f(\cdot) \equiv 0$ implies that

$$\tfrac{d}{dt} f = 2x(t) \cdot \dot{x}(t) + 2y(t) \cdot \dot{y}(t) \equiv 0, \quad \forall\, t.$$

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The reaction force $R(\cdot)$ cannot change the energy of the system

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Therefore $$R_x = \lambda \cdot x, \quad R_y = \lambda \cdot y.$$

For defining λ consider the 2^{nd} derivative of the constraint $f(\cdot) \equiv 0$

$$\begin{aligned} 0 \equiv \tfrac{d^2}{dt^2} f &= 2\dot{x}^2 + 2x \cdot \ddot{x} + 2\dot{y}^2 + 2y \cdot \ddot{y} \\ &= 2\dot{x}^2 + 2x \cdot \left(\tfrac{1}{m}\lambda \cdot x\right) + 2\dot{y}^2 + 2y \cdot \left(\tfrac{1}{m}\lambda \cdot y - g\right) \\ &= 2\dot{x}^2 + 2\dot{y}^2 + 2\tfrac{1}{m}\lambda \cdot (x^2 + y^2) - 2y \cdot g \end{aligned}$$

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$$R_x = \lambda \cdot x, \quad R_y = \lambda \cdot y.$$

The Lagrangian multiplier is then equal to

$$\lambda = \frac{m}{l^2}\left(y \cdot g - \dot{x}^2 - \dot{y}^2\right)$$

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The point mass dynamics in excessive coordinates (x, y) are

$$m \cdot \ddot{x} \;=\; \frac{m}{l^2}\left(y \cdot g - \dot{x}^2 - \dot{y}^2\right) \cdot x$$

$$m \cdot \ddot{y} \;=\; \frac{m}{l^2}\left(y \cdot g - \dot{x}^2 - \dot{y}^2\right) \cdot y - m \cdot g$$

Example: point-mass dynamics in generalized coordinates

To derive the point-mass dynamics with the constraint

$$f\left(x(t), y(t)\right) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall\, t$$

observe that the point's position is determined by the angle θ as

$$x(t) = l \cdot \sin\theta(t), \quad y(t) = -l \cdot \cos\theta(t).$$

The Lagrangian of the system is then

$$\begin{aligned} \mathcal{L} = K - \Pi &= \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - m \cdot g \cdot y \\ &= \tfrac{1}{2} m \cdot l^2 \cdot \dot{\theta}^2 + m \cdot g \cdot l \cdot \cos\theta \end{aligned}$$

The dynamics are

$$\begin{aligned} 0 = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] - \frac{\partial \mathcal{L}}{\partial \theta} &= m \cdot l^2 \cdot \ddot{\theta} + m \cdot g \cdot l \cdot \sin\theta \\ &= m \cdot l^2 \cdot \left(\ddot{\theta} + \frac{g}{l} \cdot \sin\theta\right), \end{aligned}$$

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Example: point-mass dynamics

The equations written in excessive coordinates (x, y)

$$\begin{array}{rcl} m \cdot \ddot{x} & = & \frac{m}{l^2}\left(y \cdot g - \dot{x}^2 - \dot{y}^2\right) \cdot x \\ m \cdot \ddot{y} & = & \frac{m}{l^2}\left(y \cdot g - \dot{x}^2 - \dot{y}^2\right) \cdot y - m \cdot g \end{array} \tag{1}$$

and the equation written in generalized coordinate θ

$$\ddot{\theta} + \frac{g}{l} \cdot \sin\theta = 0 \tag{2}$$

represent the dynamics of the same system provided that the initial conditions of both differential equations are appropriately chosen.

However, for mechanical systems with constraints

- the equations of the form (1) can be always derived,
- while the equations of the form (2) might not.

Example: point-mass dynamics

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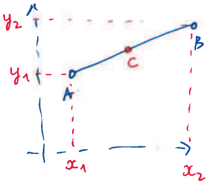
Systems with non-holonomic constraints

Example: constrained two point-masses dynamics

Consider two point masses of $m = 1$ [kg] each connected by massless rod of length l and moving in the vertical plane.



Constraint No. 1: $\Big(x_1(t) - x_2(t)\Big)^2 + \Big(y_1(t) - y_2(t)\Big)^2 = l^2, \; \forall\, t$

Constraint No. 2: Assume that the velocity of the center of the rod – point C on the plot

$$\vec{v}_C = \left[\tfrac{1}{2}(\dot{x}_1 + \dot{x}_2);\; \tfrac{1}{2}(\dot{y}_1 + \dot{y}_2)\right]$$

always aligned with the rod.

$$\Big(x_2(t) - x_1(t)\Big)\Big(\dot{y}_1(t) + \dot{y}_2(t)\Big) - \Big(y_2(t) - y_1(t)\Big)\Big(\dot{x}_1(t) + \dot{x}_2(t)\Big) = 0$$

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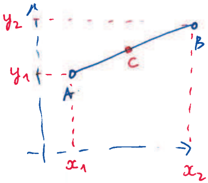

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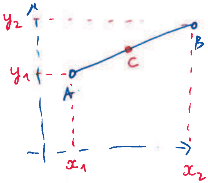

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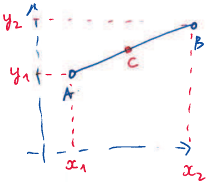

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Example: constrained two point-masses dynamics

The dynamics of the system with coordinates (x_1, y_1, x_2, y_2) are

$$\ddot{x}_1 = R^{(1)}_{x_1} + R^{(2)}_{x_1} \qquad \ddot{x}_2 = R^{(1)}_{x_2} + R^{(2)}_{x_2}$$

$$\ddot{y}_1 = R^{(1)}_{y_1} + R^{(2)}_{y_1} - g \qquad \ddot{y}_2 = R^{(1)}_{y_2} + R^{(2)}_{y_2} - g$$

Here $R^{(1)}$ and $R^{(2)}$ are the reaction forces due to constraints

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Components of of the reaction forces are determined from the assumption that such forces do not dissipate or increase the energy of the system along its motions

$$R^{(i)}_{x_1} \cdot \dot{x}_1 + R^{(i)}_{y_1} \cdot \dot{y}_1 + R^{(i)}_{x_2} \cdot \dot{x}_2 + R^{(i)}_{y_2} \cdot \dot{y}_2 \equiv 0, \quad i = 1, 2.$$

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Observation: The system dynamics are written in 4 excessive coordinates and have 2 constraints. But one cannot reduce a number of coordinates to 2 and derive the dynamics! ☹☹☹

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Observation: The system dynamics are integrable in the sense that all the solutions can be found explicitly! ☺☺☺ see the homework!

Stability of a motion of a mechanical system

Example: a mathematical pendulum

Let us investigate a stability of equilibriums of the system

$$\ddot{\theta} + \frac{g}{l} \cdot \sin\theta = 0. \quad (3)$$

There are two equilibriums $(\bmod 2\pi)$: $\theta_e = 0$ and $\theta_e = \pi$.

Linearization of the dynamics at $\theta_e = \pi$ results in

$$\ddot{z} - \frac{g}{l} \cdot z = 0.$$

The equilibrium of this linear system is unstable. Therefore, the equilibrium $\theta_e = \pi$ of the nonlinear system (3) is unstable as well.

Linearization of the dynamics at $\theta_e = 0$ results in

$$\ddot{z} + \frac{g}{l} \cdot z = 0.$$

This linear system has the center at the origin and the nonlinear dynamics (3) has the first integral. Therefore, the nonlinear system (3) has the center at $\theta_e = 0$ and this equilibrium is stable!

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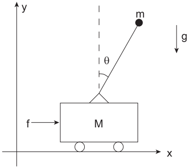
Linearization of the dynamics at $\theta_e = 0$ results in

$$\ddot{z} + \frac{g}{l} \cdot z = 0.$$

This linear system has the center at the origin and the nonlinear dynamics (3) has the first integral. Therefore, the nonlinear system (3) has the center at $\theta_e = 0$ and this equilibrium is stable!

Example: a pendulum on a cart

Consider a pendulum (a point of a mass m at the distance l from the suspension point) attached to a cart of a mass M, which freely moves on the horizontal with $f = 0$.



When $M = m = l = 1$ the dynamics in coordinates (x, θ) are

$$\begin{aligned} 2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 &= 0 \; (= f) \\ \cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta &= 0 \end{aligned}$$

and have two sets of equlibriums:

- the pendulum is up, the cart is in any position: $\theta_e = 0$, $\forall\, x_e$
- the pendulum is down, the cart is in any position: $\theta_e = \pi$, $\forall\, x_e$

Let us investigate their stability!

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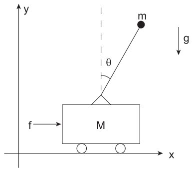

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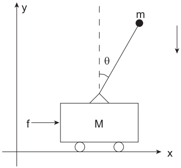

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Let us investigate their stability!

Example: a pendulum on a cart

The system
$$\begin{aligned} 2\cdot\ddot{x} + \cos\theta\cdot\ddot{\theta} - \sin\theta\cdot\dot{\theta}^2 &= 0 \\ \cos\theta\cdot\ddot{x} + \ddot{\theta} - g\cdot\sin\theta &= 0 \end{aligned}$$

has two integrals of motion (CoM): the total energy

$$E = \frac{1}{2}\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}^T \begin{bmatrix} 2 & \cos\theta \\ \cos\theta & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + g\cdot\cos\theta$$

and the momentum conjugated to the x-coordinate

$$P(x(t),\theta(t),\dot{x}(t),\dot{\theta}(t)) = 2\cdot\dot{x}(t) + \cos\theta(t)\cdot\dot{\theta}(t)$$

The last relation can be integrated again and leads to

$$\begin{aligned} x(t) &= x(0) + \tfrac{1}{2}\sin\theta(0) - \tfrac{1}{2}\sin\theta(t) + \tfrac{1}{2}P(x(0),\theta(0),\dot{x}(0),\dot{\theta}(0))\cdot t \\ &= -\tfrac{1}{2}\sin\theta(t) + C_0 + C_1\cdot t \end{aligned}$$

If one is able to compute $\theta(\cdot)$, then the formula gives $x(\cdot)$!

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Example: a pendulum on a cart

For decoupling dynamics of the θ-variable in the system

$$\begin{aligned} 2\cdot\ddot{x} + \cos\theta\cdot\ddot{\theta} - \sin\theta\cdot\dot{\theta}^2 &= 0 \\ \cos\theta\cdot\ddot{x} + \ddot{\theta} - g\cdot\sin\theta &= 0 \end{aligned}$$

one can use the second equation.

Indeed

$$\cos\theta\cdot\overbrace{\frac{1}{2}\left(\sin\theta\cdot\dot{\theta}^2 - \cos\theta\cdot\ddot{\theta}\right)}^{=\ddot{x}} + \ddot{\theta} - g\cdot\sin\theta = 0$$

substituting and collecting the terms one derives the equation

$$\left(1 - \frac{1}{2}\cos^2\theta\right)\cdot\ddot{\theta} + \frac{1}{2}\cos\theta\cdot\sin\theta\cdot\dot{\theta}^2 - g\cdot\sin\theta = 0$$

What can one say on the dynamics of this system
in a vicinity of its equilibriums at 0 and at π?

Example: a pendulum on a cart

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The linearization of the system

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in a vicinity of the equilibrium $\theta_e = 0$ is unstable

$$\ddot{z} + \left[\frac{d}{d\theta}\left(\frac{-g\cdot\sin\theta}{1-\frac{1}{2}\cos^2\theta}\right)\right]\Bigg|_{\theta=0}\cdot z = \ddot{z} - 2\cdot g\cdot z = 0$$

Its linearization at $\theta_e = \pi$ is stable

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In addition the system has the first integral

$$E_{red} = \tfrac{1}{2}\left(1 - \tfrac{1}{2}\cos^2\theta\right)\cdot\dot{\theta}^2 + g\cdot\cos\theta$$

The nonlinear system has the center at $\theta_e = \pi$!

Example: a pendulum on a cart

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Example: a pendulum on a cart

To conclude:

- Any cart pendulum solution can be written as

$x(t) = \frac{1}{2}\sin\theta(t) + C_0 + C_1 t$ and $\theta(t)$ is a solution of reduced dynamics

- In a vicinity of the upright equilibrium $[\theta_e = 0,\, x = x_e]$ the reduced dynamics is hyperbolic, therefore any of upright equilibriums is unstable.
- In a vicinity of the downward equilibrium $[\theta_e = \pi,\, x = x_e]$ the reduced dynamics is stable, but $x(t)$ will drift with $C_1 \neq 0$. Hence any of downward equilibriums is unstable as well.

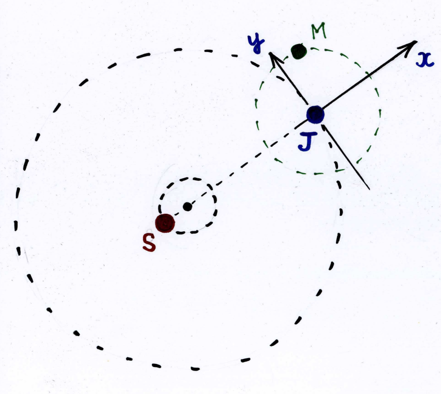
Lagrange-Dirichlet theorem

Theorem (1788)

If at the position of an isolated equilibrium of a conservative mechanical system with holonomic constraints the potential energy Π has a strict minimum, then this equilibrium is stable.

Equations of motion for the position of the Moon in rotating coordinate frame are

$$\begin{cases} \ddot{x} - 2m\dot{y} &= \frac{\partial}{\partial x} F \\ \ddot{y} + 2m\dot{x} &= \frac{\partial}{\partial y} F \end{cases}$$



Here

$$F = \frac{\kappa}{\sqrt{x^2 + y^2}} + \frac{3}{2} m^2 x^2$$

m, κ are positive constants.

The system has the invariant: $I = \dot{x}^2 + \dot{y}^2 - 2F(x, y) + C$

Task: Analyze the dynamics in a vicinity of the periodic motion

Example: restricted 3 body problem

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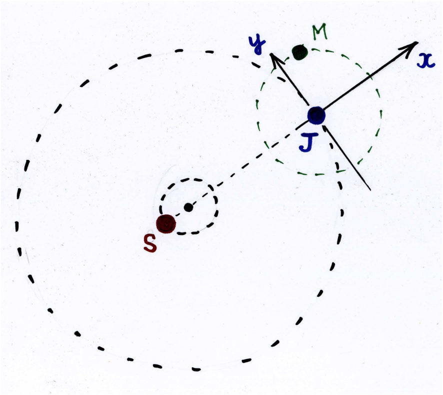

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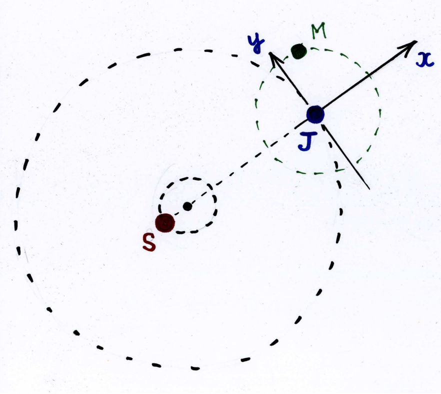

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Task: Analyze the dynamics in a vicinity of the periodic motion

Example: elements of theory of G.W. Hill

Denote as $[\, x_p(t),\ y_p(t)\,]$ the periodic solution

Perturbed solutions $[x_p(t) + \delta x(t),\ y_p(t) + \delta y(t)]$ defined by

$$\frac{d^2}{dt^2}\left[\delta x\right] - 2m\frac{d}{dt}\left[\delta y\right] = \\ = \left[\frac{\partial^2}{\partial x^2}F(x_p(t), y_p(t))\right]\delta x + \left[\frac{\partial^2}{\partial x \partial y}F(x_p(t), y_p(t))\right]\delta y$$

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The integral Jacobi $I(\cdot)$ gives another relation

$$\frac{d}{dt}x_p(t)\frac{d}{dt}\left[\delta x\right] + \frac{d}{dt}y_p(t)\frac{d}{dt}\left[\delta y\right] = \\ = \left[\frac{\partial}{\partial x}F(x_p(t), y_p(t))\right]\delta x + \left[\frac{\partial}{\partial y}F(x_p(t), y_p(t))\right]\delta y$$

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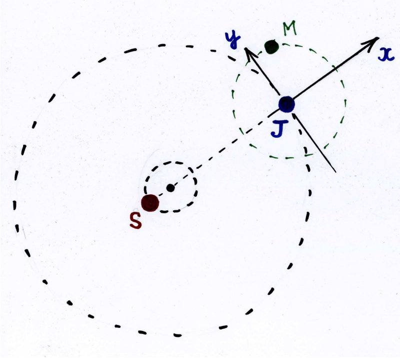

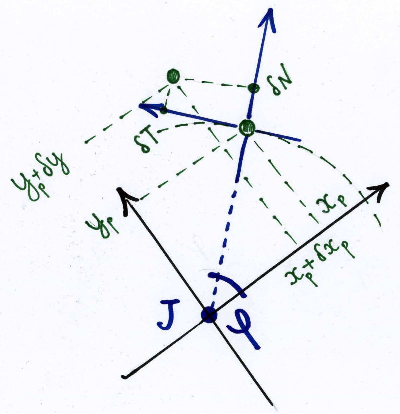

Transform of coordinates into normal (δN) and tangent (δT)

$$\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \delta T \\ \delta N \end{bmatrix}$$

Example: elements of theory of G.W. Hill

In a vicinity of the motion the original coordinates

$$\left[x,\, y,\, \dot{x},\, \dot{y}\right]$$

are changed into

$$\left[\phi,\, I,\, N,\, \dot{N}\right]$$

The linearization of $\phi(\cdot)$ is not important: it perpetually rotates

The linearization of $I(\cdot)$ is straightforward: $\frac{d}{dt}\left[\delta I\right] = 0$

The linearization of $[N, \dot{N}]$ is the famous Hill's equation

$$\frac{d^2}{dt^2}\left[\delta N\right] + \Phi(t)\delta N = 0$$

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Outcomes of the example

Analysis of dynamics in a vicinity of the motion's orbit requires:

- Decomposition of coordinates into
  - transverse to the trajectory ($\dim = 2n - 1$)
  - along the trajectory ($\dim = 1$)

  In the example they are

$$\left[I,\ N,\ \dot{N} \right] \quad \text{and} \quad \phi$$

Outcomes of the example

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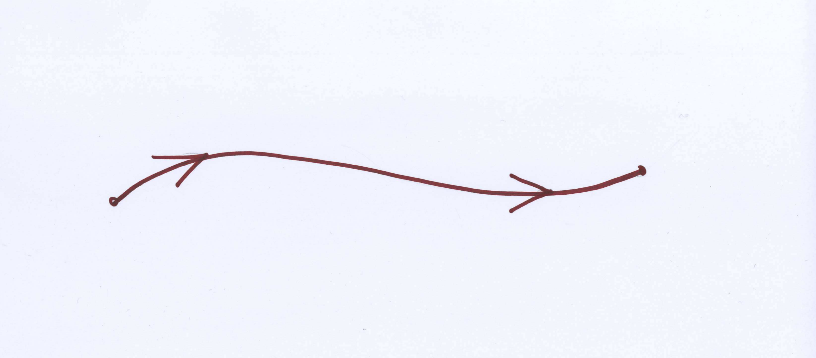
  In the example they are

$$\left[I,\ N,\ \dot{N}\right] \quad \text{and} \quad \phi$$

- Presence of invariants allows to reduce a number of transverse coordinates with non-trivial dynamics.

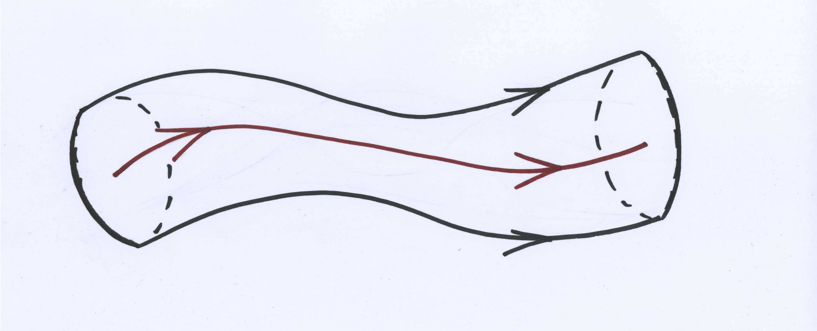
  In the example the integral Jacobi $I(\cdot)$ is excluded.

Outcomes of the example: geometrical interpretation



Given a trajectory of a nominal motion

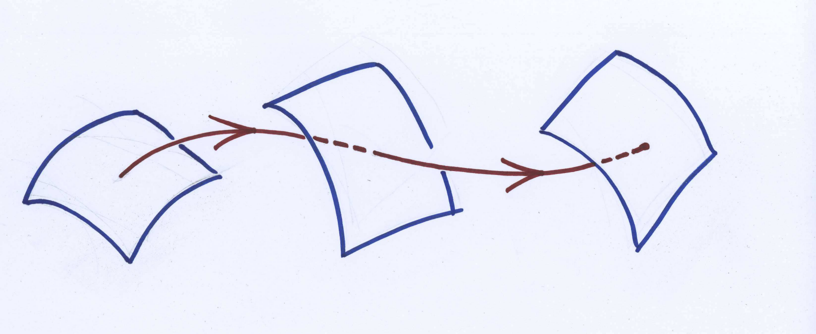
Outcomes of the example: geometrical interpretation



We would like to analyze properties of the dynamics

in its tubing vicinity

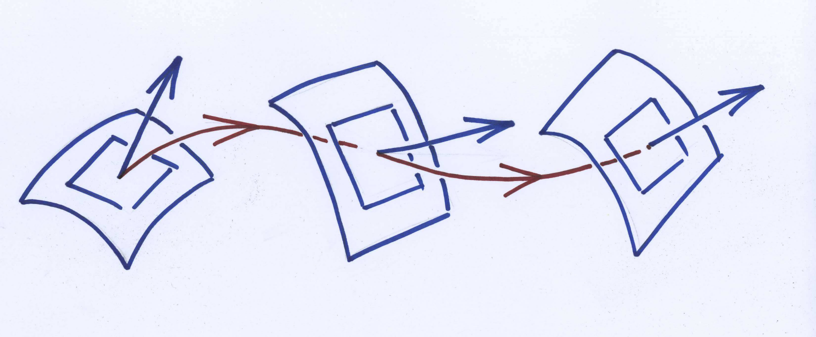
Outcomes of the example: geometrical interpretation



Introduce a family of dis-joint transverse surfaces

that are continuously slicing this vicinity

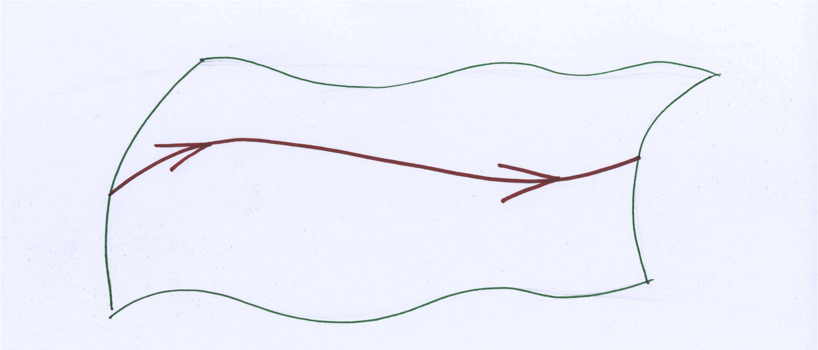
Outcomes of the example: geometrical interpretation



For the linearization of the dynamics the surfaces

are substituted by tangent planes

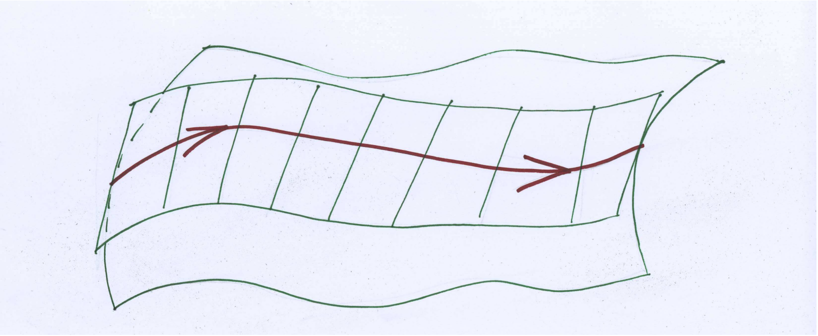
Outcomes of the example: geometrical interpretation



If the dynamics have some invariants,

then they define a manifold

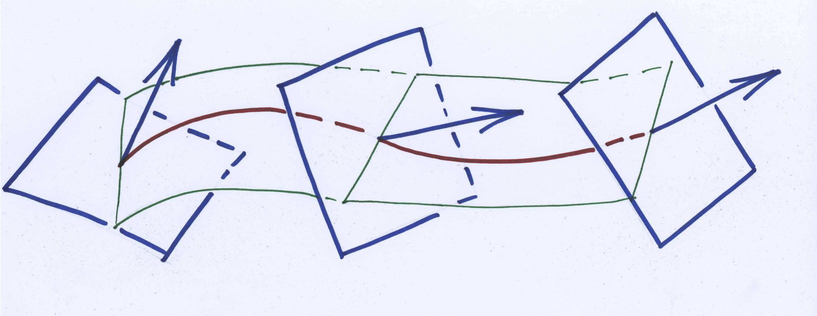
Outcomes of the example: geometrical interpretation



For the linearization we consider the linear subspaces

that are tangent to to the trajectory along this manifold

Outcomes of the example: geometrical interpretation



Evolution of coordinates on these linear subspaces will define

linearization of transverse coordinates with nontrivial behavior