

Lecture 4

Cases Studies in Motion Planning for Underactuated Systems

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Learning outcomes: Problem formulation and settings for searching and planning behaviors of mechanical systems with one or two passive degrees of freedom. Examples

1. Example: a pendulum on a cart

1.1 Moving a pendulum over an obstacle (a wall)

2. Example: a coin rolling on a table

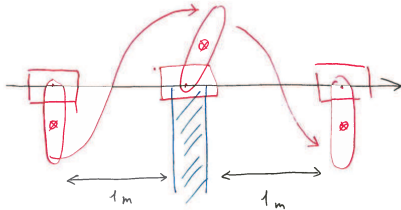
2.1 Planning a rolling of a coin of along a circle

3. Example: a spherical pendulum on a puck

3.1 Motions when a passive sph. pendulum stays above the horizontal

Example: a pendulum on a cart

Cart-pendulum system: moving a pendulum over a wall

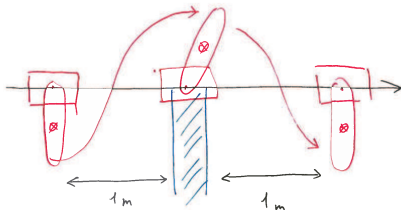


The task is to find a force applied to the cart such that in response the pendulum of the system comes over a wall without collision

One of conceptual solutions:

1. Push the cart to left to swing up the pendulum
2. Push the cart to right to pass the wall keeping the pendulum above the horizontal
3. Bring the cart to the final position simultaneously damping the pendulum oscillations

Cart-pendulum system: moving a pendulum over a wall



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For the step 1 the dynamics with constant force $f_1 < 0$ are

$$2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 = f_1$$

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

f_1 should be sufficient to swing the pendulum above the horizontal

To find f_1 , one needs to decouple the dynamics of x and θ

E.g. one can take \ddot{x} from the 1st eqn and plug it into the 2nd eqn

$$\cos \theta \cdot \frac{1}{2} \left(\underbrace{f_1 - \cos \theta \cdot \ddot{\theta} + \sin \theta \cdot \dot{\theta}^2}_{= \ddot{x}} \right) + \ddot{\theta} - g \cdot \sin \theta = 0$$

Collecting the terms results in the dynamics of θ -variable

$$\left(1 - \frac{1}{2} \cos^2 \theta\right) \cdot \ddot{\theta} + \frac{1}{2} \cos \theta \cdot \sin \theta \cdot \dot{\theta}^2 + \left(\frac{1}{2} \cdot f_1 \cdot \cos \theta - g \cdot \sin \theta\right) = 0$$

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Cart-pendulum system: moving a pendulum over a wall

The initial conditions for a searched motion $\theta^*(\cdot)$ of the system

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are defined as

$$\theta^*(0) = \pi, \quad \dot{\theta}^*(0) = 0$$

How to find f_1 such that the solution $\theta^*(\cdot)$ will pass through

$$\theta^*(T) = a, \quad \dot{\theta}^*(T) = 0 \quad ???$$

The θ -dynamics has the integral of motion for any f_1 defined as

$$E_{red}(\theta(t), \dot{\theta}(t)) = \frac{1}{2} \left(1 - \frac{1}{2} \cos^2 \theta(t)\right) \dot{\theta}^2(t) + g \cos \theta(t) + \frac{1}{2} f_1 \sin \theta(t) \equiv C$$

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Hence

$$\begin{aligned} E_{red}(\theta^*(T), \dot{\theta}^*(T)) &= E_{red}(a, 0) = g \cdot \cos a + \frac{1}{2} \cdot f_1 \cdot \sin a \\ &= E_{red}(\theta^*(0), \dot{\theta}^*(0)) = g \cdot \cos \pi + \frac{1}{2} \cdot f_1 \cdot \sin \pi \end{aligned}$$

Cart-pendulum system: moving a pendulum over a wall

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Cart-pendulum system: moving a pendulum over a wall

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Hence

$$g \cdot \cos a + \frac{1}{2} \cdot f_1 \cdot \sin a = -g$$

$$f_1 = -2 \cdot \frac{1 + \cos a}{\sin a} \cdot g$$

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How to compute a time constant T?

Cart-pendulum system: moving a pendulum over a wall

To estimate the position and velocity of the cart at the end of this maneuver, one can observe that

$$\begin{aligned}2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 &= f_1 \\ &\Downarrow \\ \frac{d}{dt} [2 \cdot \dot{x} + \cos \theta \cdot \dot{\theta}] &= f_1 \\ &\Downarrow \\ \{2 \cdot \dot{x}(t) + \cos \theta(t) \cdot \dot{\theta}(t)\} - \{2 \cdot \dot{x}(0) + \cos \theta(0) \cdot \dot{\theta}(0)\} &= f_1 \cdot t \\ &\Downarrow \\ \{2 \cdot x(t) + \sin \theta(t)\} - \{2 \cdot x(0) + \sin \theta(0)\} - \\ \{2 \cdot \dot{x}(0) + \cos \theta(0) \cdot \dot{\theta}(0)\} \cdot t &= \frac{1}{2} \cdot f_1 \cdot t^2\end{aligned}$$

If the system was at rest in the beginning of the movement, then

$$2x(T) + \sin a - 2x(0) = \frac{1}{2} \cdot f_1 \cdot T^2, \quad 2\dot{x}(T) = f_1 \cdot T$$

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Cart-pendulum system: moving a pendulum over a wall

On the 2nd step, one needs to find a constant $f_2 > 0$ such that for the solution of the system

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the position of the cart moves behind the wall while the pendulum stays above the horizontal for the same time interval.

As before the behavior of the θ -variable is defined by as a solution $\theta^*(\cdot)$ of the reduced dynamics

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with initial conditions at

$$\theta^*(0) = a, \quad \dot{\theta}^*(0) = 0 \quad \text{with} \quad -\frac{\pi}{2} < a < \frac{\pi}{2}$$

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As for the 1st maneuver, one can use the integral of θ -dynamics

$$E_{red}(\theta(t), \dot{\theta}(t)) = \frac{1}{2} \left(1 - \frac{1}{2} \cos^2 \theta(t) \right) \dot{\theta}^2(t) + g \cos \theta(t) + \frac{1}{2} f_2 \sin \theta(t) \equiv C$$

for computing for the second maneuver

- time to reach the critical values of the angle: $\theta_c = \pm \frac{\pi}{2}$
- angular velocity of the pendulum at critical values of θ

as functions of the input variable f_2 .

Homework: find constants f_1 and f_2 , which in combination create the requested behavior of the cart-pendulum.

One can investigate more complicated situations when

$$f_1 = f_{10} + f_{11} \cdot \theta + \dots + f_{1k} \cdot \theta^k, \quad f_2 = f_{20} + f_{21} \cdot \theta + \dots + f_{2m} \cdot \theta^m$$

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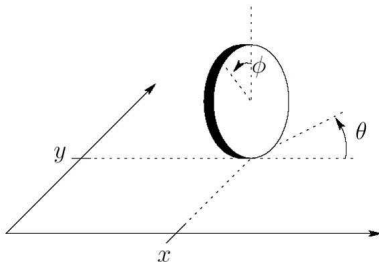
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A coin rolling on a table

Example: planning a rolling of a coin



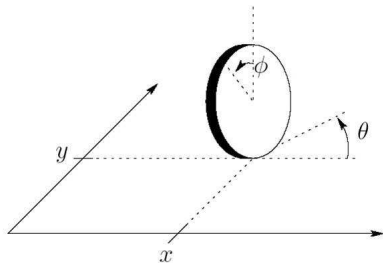
The equations of motion are

$$m\ddot{x} = F_x^c, \quad m\ddot{y} = F_y^c, \quad J\ddot{\theta} = u$$

Here u is control; $F^c = [F_x^c; F_y^c]$ is the force due to the constraint

$$\dot{y}(t) \cos \theta(t) - \dot{x}(t) \sin \theta(t) \equiv 0, \quad \forall t$$

Example: planning a rolling of a coin

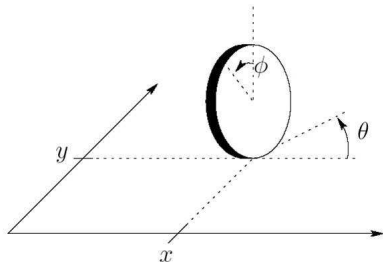


The equations of motion are

$$m\ddot{x} = \lambda \cdot \cos\left(\theta - \frac{\pi}{2}\right), \quad m\ddot{y} = \lambda \cdot \sin\left(\theta - \frac{\pi}{2}\right), \quad J\ddot{\theta} = u$$

Here λ is amplitude of the constraint force.

Example: planning a rolling of a coin

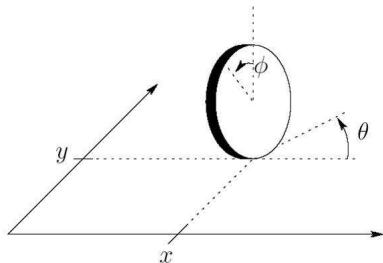


The equations of motion are

$$\begin{aligned}\ddot{x} &= -[\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \sin(\theta) \\ \ddot{y} &= [\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \cos(\theta) \\ J\ddot{\theta} &= u\end{aligned}$$

where constraint force is quadratic in velocities.

Example: planning a rolling of a coin



We have already seen irreducible equations of motion of the form

$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}} \mathcal{L} \right] - \frac{\partial}{\partial q} \mathcal{L} = R(q, \dot{q}) + B(q) \mathbf{u}, \quad R_i = \dot{q}^T r_i(q) \dot{q}$$

Here $q \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, and $R(\cdot)$ is a vector of reaction forces.

Example: planning a rolling of a coin of along a circle

Motion planning for the dynamical model

$$\begin{aligned}\ddot{x} &= -[\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \sin(\theta) \\ \ddot{y} &= [\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \cos(\theta) \\ J\ddot{\theta} &= u\end{aligned}$$

can be quite non-trivial.

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Find feasible motions of the system consistent with requirement
the center of mass should stay on a circle of radius R

Example: planning a rolling of a coin of along a circle

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I.e. along any such motion $[x_c(t), y_c(t), \theta_c(t)]$ the relations hold

$$x_c(t) = R \cdot \cos\left(\theta_c(t) - \frac{\pi}{2}\right)$$

$$y_c(t) = R \cdot \sin\left(\theta_c(t) - \frac{\pi}{2}\right)$$

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$$y_c(t) = -R \cdot \cos \theta_c(t)$$

$$\dot{x}_c(t) = R \cdot \cos \theta_c(t) \cdot \dot{\theta}_c(t)$$

$$\dot{y}_c(t) = R \cdot \sin \theta_c(t) \cdot \dot{\theta}_c(t)$$

Example: planning a rolling of a coin of along a circle

Motion planning for the dynamical model

$$\begin{aligned}\ddot{x} &= -[\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \sin(\theta) \\ \ddot{y} &= [\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \cos(\theta) \\ J\ddot{\theta} &= \mathbf{u}\end{aligned}$$

can be quite non-trivial.

I.e. along any such motion $[x_c(t), y_c(t), \theta_c(t)]$ the relations hold

$$x_c(t) = R \cdot \sin \theta_c(t)$$

$$y_c(t) = -R \cdot \cos \theta_c(t)$$

$$\dot{x}_c(t) = R \cdot \cos \theta_c(t) \cdot \dot{\theta}_c(t)$$

$$\dot{y}_c(t) = R \cdot \sin \theta_c(t) \cdot \dot{\theta}_c(t)$$

$$\ddot{x}_c = R \left[\cos \theta_c \ddot{\theta}_c - \sin \theta_c \dot{\theta}_c^2 \right]$$

$$\ddot{y}_c = R \left[\sin \theta_c \ddot{\theta}_c + \cos \theta_c \dot{\theta}_c^2 \right]$$

Example: planning a rolling of a coin of along a circle

Along a circular motion $[x_c(t), y_c(t), \theta_c(t)]$ of the system

$$\ddot{x} = -[\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \sin(\theta)$$

$$\ddot{y} = [\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \cos(\theta)$$

$$J\ddot{\theta} = u$$

the relations hold

$$\ddot{x}_c = R \left[\cos \theta_c \ddot{\theta}_c - \sin \theta_c \dot{\theta}_c^2 \right]$$

$$\ddot{y}_c = R \left[\sin \theta_c \ddot{\theta}_c + \cos \theta_c \dot{\theta}_c^2 \right]$$

Example: planning a rolling of a coin of along a circle

Along a circular motion $[x_c(t), y_c(t), \theta_c(t)]$ of the system

$$\ddot{x} = -[\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \sin(\theta)$$

$$\ddot{y} = [\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \cos(\theta)$$

$$J\ddot{\theta} = u$$

the relations hold

$$\cos \theta_c \cdot \ddot{x}_c = \cos \theta_c \cdot R \cdot [\cos \theta_c \ddot{\theta}_c - \sin \theta_c \dot{\theta}_c^2]$$

$$\sin \theta_c \cdot \ddot{y}_c = \sin \theta_c \cdot R \cdot [\sin \theta_c \ddot{\theta}_c + \cos \theta_c \dot{\theta}_c^2]$$

Example: planning a rolling of a coin of along a circle

Along a circular motion $[x_c(t), y_c(t), \theta_c(t)]$ of the system

$$\ddot{x} = -[\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \sin(\theta)$$

$$\ddot{y} = [\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \cos(\theta)$$

$$J\ddot{\theta} = \mathbf{u}$$

the relations hold

$$\cos \theta_c \cdot \ddot{x}_c = \cos \theta_c \cdot R \cdot [\cos \theta_c \ddot{\theta}_c - \sin \theta_c \dot{\theta}_c^2]$$

$$\sin \theta_c \cdot \ddot{y}_c = \sin \theta_c \cdot R \cdot [\sin \theta_c \ddot{\theta}_c + \cos \theta_c \dot{\theta}_c^2]$$

↓

$$\cos \theta_c \cdot \ddot{x}_c + \sin \theta_c \cdot \ddot{y}_c = R\ddot{\theta}_c$$

Example: planning a rolling of a coin of along a circle

Along a circular motion $[x_c(t), y_c(t), \theta_c(t)]$ of the system

$$\ddot{x} = -[\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \sin(\theta)$$

$$\ddot{y} = [\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \cos(\theta)$$

$$J\ddot{\theta} = u$$

the relations hold

$$\cos \theta_c \cdot \ddot{x}_c = \cos \theta_c \cdot R \cdot [\cos \theta_c \ddot{\theta}_c - \sin \theta_c \dot{\theta}_c^2]$$

$$\sin \theta_c \cdot \ddot{y}_c = \sin \theta_c \cdot R \cdot [\sin \theta_c \ddot{\theta}_c + \cos \theta_c \dot{\theta}_c^2]$$

\Downarrow

$$\cos \theta_c \cdot \ddot{x}_c + \sin \theta_c \cdot \ddot{y}_c = R \cdot \ddot{\theta}_c$$

\Downarrow

$$0 = R \cdot \ddot{\theta}_c$$

Example: planning a rolling of a coin of along a circle

Any circular motion $[x_c(t), y_c(t), \theta_c(t)]$ of the system

$$\ddot{x} = -[\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \sin(\theta)$$

$$\ddot{y} = [\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \cos(\theta)$$

$$J\ddot{\theta} = u$$

has the form

$$\theta_c(t) = \omega_c \cdot t + \theta_0$$

$$x_c(t) = R \cdot \sin \theta_c(t)$$

$$y_c(t) = -R \cdot \cos \theta_c(t)$$

$$u_c(t) = 0$$

How to find new coordinates for the dynamics that are zero on the nominal motion and nonzero away from it? Regulating them to zero helps stabilizing such motion **orbitally**

Example: planning a rolling of a coin of along a circle

Any circular motion $[x_c(t), y_c(t), \theta_c(t)]$ of the system

$$\ddot{x} = -[\dot{y} \cdot \sin \theta + \dot{x} \cdot \cos \theta] \cdot \dot{\theta} \cdot \sin(\theta)$$

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has the form

$$\theta_c(t) = \omega_c \cdot t + \theta_0$$

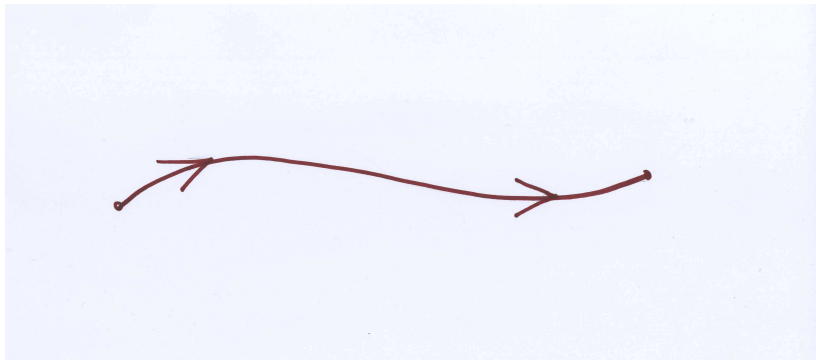
$$x_c(t) = R \cdot \sin \theta_c(t)$$

$$y_c(t) = -R \cdot \cos \theta_c(t)$$

$$u_c(t) = 0$$

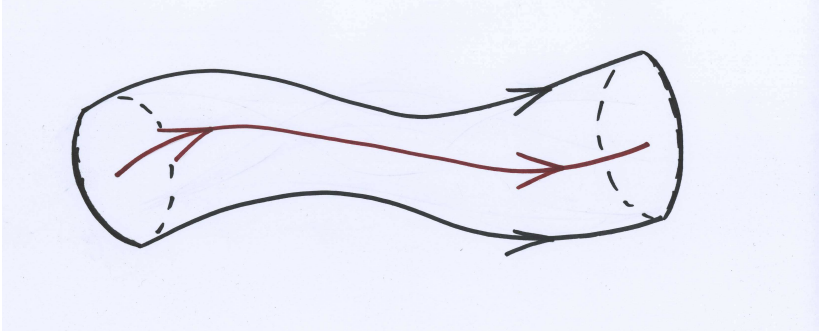
How to find new coordinates for the dynamics that are zero on the nominal motion and nonzero away from it? Regulating them to zero helps stabilizing such motion **orbitally**

Geometrical interpretation



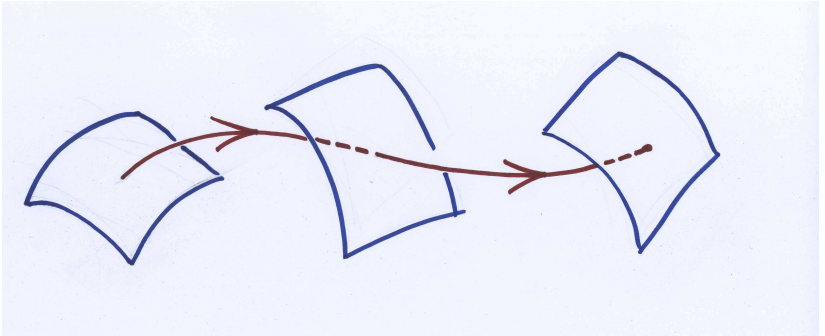
Given a trajectory of a nominal motion

Geometrical interpretation



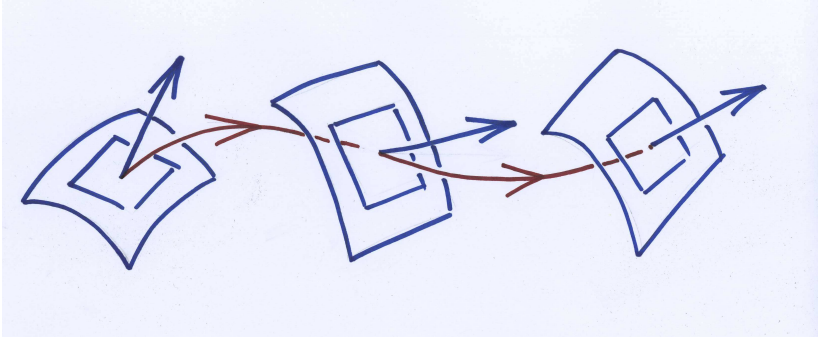
We would like to analyze properties of the dynamics
in its tubing vicinity

Geometrical interpretation



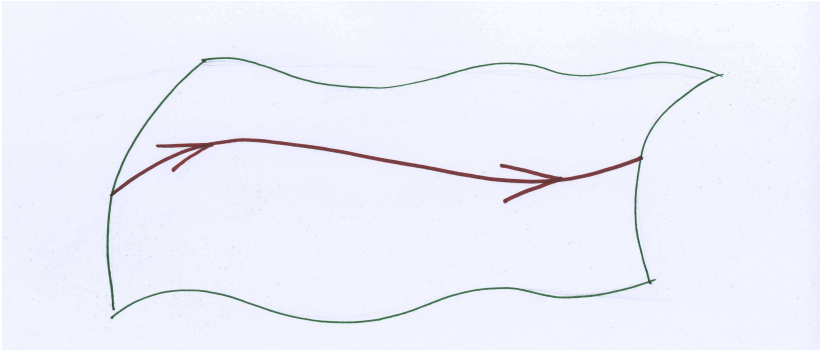
Introduce a family of dis-joint transverse surfaces
that are continuously slicing this vicinity

Geometrical interpretation



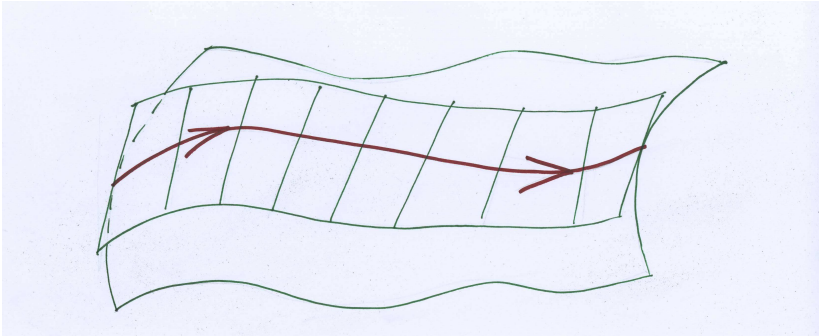
For the linearization of the dynamics the surfaces
are substituted by tangent planes

Geometrical interpretation



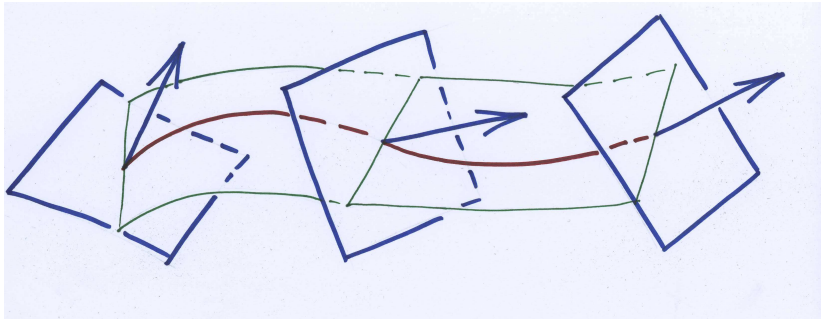
If the dynamics have some invariants,
then they define a manifold

Geometrical interpretation



For the linearization we consider the linear subspaces that are tangent to the trajectory along this manifold

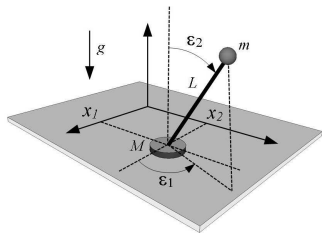
Geometrical interpretation



Evolution of coordinates on these linear subspaces will define linearization of transverse coordinates with nontrivial behavior

Example: a spherical pendulum on a puck

Example: motions of a spherical pendulum on a puck



The system has four generalized coordinates: (x_1, x_2) are variables for representing the position of the puck on a plane; (ϵ_1, ϵ_2) are two angles (precession and nutation) for representing the status of the pendulum.

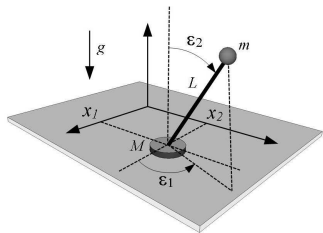
The dynamics of the system are

$$\begin{aligned} \frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{\epsilon}_1} \right] - \frac{\partial \mathcal{L}}{\partial \epsilon_1} &= 0 & \frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{\epsilon}_2} \right] - \frac{\partial \mathcal{L}}{\partial \epsilon_2} &= 0 \\ \frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \right] - \frac{\partial \mathcal{L}}{\partial x_1} &= F_1 & \frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{x}_2} \right] - \frac{\partial \mathcal{L}}{\partial x_2} &= F_2 \end{aligned}$$

with F_1, F_2 being external forces acting on the puck.

The task: to find behaviors of the system when the puck is forced to move along of a circle of a radius R

Example: motions of a spherical pendulum on a puck



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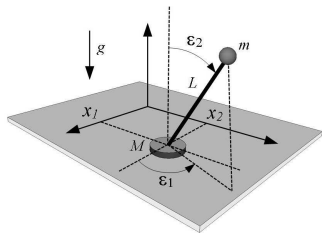
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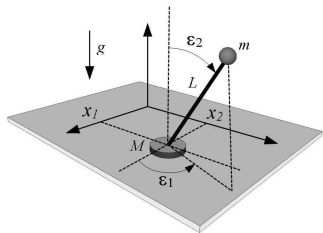
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Example: motions of a spherical pendulum on a puck



The task means that for any searched behavior in polar coordinates on a plane

$$x_1(t) = R \cdot \cos \psi(t)$$

$$x_2(t) = R \cdot \sin \psi(t)$$

while the behaviors of $\varepsilon_1(\cdot)$ and $\varepsilon_2(\cdot)$ are

$$\varepsilon_1(t) = \phi_1(\psi(t)), \quad \varepsilon_2(t) = \phi_2(\psi(t))$$

with some functions $\phi_1(\cdot)$, $\phi_2(\cdot)$.

Suppose that the angle ε_1 can be chosen as a motion generator and on a particular behavior

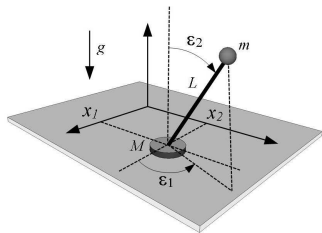
$$q^*(t) = [x_1^*(t), x_2^*(t), \varepsilon_1^*(t), \varepsilon_2^*(t)]$$

the relation between $\psi(\cdot)$ and $\varepsilon_1^*(\cdot)$ is

$$\psi(t) = \varepsilon_1^*(t).$$

What are then the equations of motion for $q^*(\cdot)$?

Example: motions of a spherical pendulum on a puck



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Example: motions of a spherical pendulum on a puck

The assumption means that

$$x_1^*(t) = R \cdot \cos \varepsilon_1^*(t)$$

$$x_2^*(t) = R \cdot \sin \varepsilon_1^*(t)$$

Furthermore it means that the equations of motion

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{\varepsilon}_1} \right] - \frac{\partial \mathcal{L}}{\partial \varepsilon_1} = 0 \quad \frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{\varepsilon}_2} \right] - \frac{\partial \mathcal{L}}{\partial \varepsilon_2} = 0$$

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \right] - \frac{\partial \mathcal{L}}{\partial x_1} = F_1 \quad \frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{x}_2} \right] - \frac{\partial \mathcal{L}}{\partial x_2} = F_2$$

for this behavior become

$$\left(R + L \cdot \sin(\varepsilon_2) \right) \ddot{\varepsilon}_1 + 2 \cdot L \cdot \cos(\varepsilon_2) \cdot \dot{\varepsilon}_1 \cdot \dot{\varepsilon}_2 = 0$$

$$L \cdot \ddot{\varepsilon}_2 - \cos(\varepsilon_2) \cdot \left(R + L \cdot \sin(\varepsilon_2) \right) \dot{\varepsilon}_1^2 - g \cdot \sin(\varepsilon_2) = 0$$

How to compute $\varepsilon_1^*(\cdot)$ and $\varepsilon_2^*(\cdot)$?

Example: motions of a spherical pendulum on a puck

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How to compute $\varepsilon_1^*(\cdot)$ and $\varepsilon_2^*(\cdot)$?

Example: motions of a spherical pendulum on a puck

The first equation of the system

$$\left(R + L \cdot \sin(\varepsilon_2)\right) \ddot{\varepsilon}_1 + 2 \cdot L \cdot \cos(\varepsilon_2) \cdot \dot{\varepsilon}_1 \cdot \dot{\varepsilon}_2 = 0$$

$$L \cdot \ddot{\varepsilon}_2 - \cos(\varepsilon_2) \cdot \left(R + L \cdot \sin(\varepsilon_2)\right) \dot{\varepsilon}_1^2 - g \cdot \sin(\varepsilon_2) = 0$$

can be written as

$$\frac{d}{dt} \left[\left(R + L \cdot \sin(\varepsilon_2)\right)^2 \dot{\varepsilon}_1 \right] = 0$$

$$\left[R + L \cdot \sin(\varepsilon_2(t))\right]^2 \dot{\varepsilon}_1(t) = \left[R + L \cdot \sin(\varepsilon_2(0))\right]^2 \dot{\varepsilon}_1(0) = C$$

Therefore, the second equation can be rewritten as

$$L \cdot \ddot{\varepsilon}_2 - \cos(\varepsilon_2) \cdot \frac{C}{\left(R + L \cdot \sin(\varepsilon_2)\right)^3} - g \cdot \sin(\varepsilon_2) = 0$$

Example: motions of a spherical pendulum on a puck

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Example: motions of a spherical pendulum on a puck

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Example: motions of a spherical pendulum on a puck

How to find solutions of the system

$$L \cdot \ddot{\varepsilon}_2 - \cos(\varepsilon_2) \cdot \frac{C}{(R + L \cdot \sin(\varepsilon_2))^3} - g \cdot \sin(\varepsilon_2) = 0 \quad ???$$

How to find the solutions that stay above the horizontal, i.e.

$$-\frac{\pi}{2} < \varepsilon_2(t) < \frac{\pi}{2} \quad \forall t \quad ???$$

The system has a conserved quantity

$$E_2 = \frac{1}{2} \dot{\varepsilon}_2^2 + \frac{g}{L} \cos(\varepsilon_2) + \frac{C^2}{2L^2 [R + L \sin(\varepsilon_2)]^2}$$

Analysis of level sets of $E_2(\cdot)$ allows answering both questions

Example: motions of a spherical pendulum on a puck

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