Lecture 5

Motion Representations for Underactuated Systems

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UNAM, Mexico
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Learning outcomes: Concepts of a motion generator (MG) and its dynamics for mechanical systems. A nested representation of motion candidates for underactuated mechanical systems. Properties of the dynamics of a MG derived based on the nested representation of a feasible behavior of an underactuated mechanical system. Examples

Outline

Example: a pendulum on a cart system

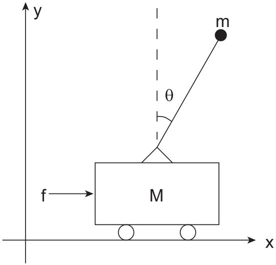
Nested representation of motions for cart-pendulum system

Given a forced motion of the system

$$[x^*(t),\, \theta^*(t),\, f^*(t)]\,,\;\; t \in [0,\, T],$$

assume that it admits alternative representation

$$x^*(t) = \phi\left(\theta^*(t)\right),\;\; t \in [0,\, T]$$



The coupled dynamics of the a pendulum on a cart are

$$\begin{aligned} 2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 &= f \\ \cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta &= \color{red}{0} \end{aligned}$$

What would be then a differential equation for describing the dynamics of θ-variable?

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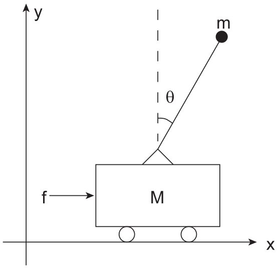

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Nested representation of motions for cart-pendulum system

The relation

$$x^*(t) = \phi\left(\theta^*(t)\right), \;\; t \in [0, T]$$

implies the corresponding relations on the 1st and 2nd derivatives

$$\dot{x}^*(t) = \phi'(\theta^*(t))\dot{\theta}^*(t), \quad \ddot{x}^*(t) = \phi''(\theta^*(t))\dot{\theta}^*(t)\dot{\theta}^*(t) + \phi'(\theta^*(t))\ddot{\theta}^*(t)$$

Substituting the relations into the non-actuated part of the dynamics

$$\cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta = 0$$

results in the decoupled dynamics for θ

$$\cos\theta \cdot \left(\phi''(\theta)\dot{\theta}^2 + \phi'(\theta)\ddot{\theta}\right) + \ddot{\theta} - g \cdot \sin\theta = 0$$

which is more transparent after collecting the similar terms

$$\left[1 + \cos\theta \cdot \phi'(\theta)\right]\ddot{\theta} + \cos\theta \cdot \phi''(\theta) \cdot \dot{\theta}^2 - g \cdot \sin\theta = 0$$

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Nested representation of motions for cart-pendulum system

The motion generator for the behavior can be chosen differently

$$\theta^*(t) = \phi\left(x^*(t)\right), \ \ t \in [0, T]$$

Similarly, it implies the relations on the derivatives

$$\dot\theta^*(t) = \phi'(x^*(t))\dot x^*(t), \quad \ddot\theta^*(t) = \phi''(x^*(t))\dot x^*(t)\dot x^*(t) + \phi'(x^*(t))\ddot x^*(t)$$

Substituting them into the non-actuated part of the dynamics

$$\cos\theta \cdot \ddot x + \ddot\theta - g \cdot \sin\theta = 0$$

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$$\cos\phi(x) \cdot \ddot x + \left(\phi''(x)\dot x^2 + \phi'(x)\ddot x\right) - g \cdot \sin\phi(x) = 0$$

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$$\left[\cos\phi(x) + \phi'(x)\right]\ddot x + \phi''(x) \cdot \dot x^2 - g \cdot \sin\phi(x) = 0$$

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In general, the motion generator for the behavior can be chosen as

$$\theta^*(t) = \phi_\theta\left(s^*(t)\right), \quad x^*(t) = \phi_x\left(s^*(t)\right) t \in [0, T]$$

Similarly, it implies the relations on the derivatives

$$\begin{aligned} \dot{\theta}^* &= \phi_\theta'(s^*)\dot{s}^*, & \ddot{\theta}^* &= \phi_\theta''(s^*)\left[\dot{s}^*\right]^2 + \phi_\theta'(s^*)\ddot{s}^* \\ \dot{x}^* &= \phi_x'(s^*)\dot{s}^*, & \ddot{x}^* &= \phi_x''(s^*)\left[\dot{s}^*\right]^2 + \phi_x'(s^*)\ddot{s}^* \end{aligned}$$

Substituting them into the non-actuated part of the dynamics

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results in the decoupled dynamics for s-variable

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What are properties of that (α-β-γ) system?

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Dynamics of the motion generator

Integrability of dynamics of the motion generator

Suppose the solution

$$\theta(t) = \theta(t, \theta_0, \dot{\theta}_0)$$

of the system

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

with initial conditions $[\theta_0, \dot{\theta}_0]$ exists.

Then the function

$$I(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0) = \dot{\theta}^2 - \psi(\theta_0, \theta)\left[\dot{\theta}_0^2 - \int_{\theta_0}^{\theta} \psi(s, \theta_0)\frac{2\gamma(s)}{\alpha(s)} ds\right]$$

with

$$\psi(\theta_0, \theta_1) = \exp\left\{-2\int_{\theta_0}^{\theta_1} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\}$$

preserves its zero-value along this (even unbounded) solution

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Example

Consider the system

$$\ddot{\theta} + \sin\theta = 0$$

that is

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

with

$$\alpha(\theta) = 1, \quad \beta(\theta) = 0, \quad \gamma(\theta) = \sin\theta$$

Then

$$I = \dot{\theta}^2 - \exp\left\{-2\int_{\theta_0}^{\theta} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\}\left[\dot{\theta}_0^2 - \int_{\theta_0}^{\theta} \exp\left\{2\int_{\theta_0}^{s} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\} \frac{2\gamma(s)}{\alpha(s)} ds\right]$$

$$= \quad \dot{\theta}^2 - \dot{\theta}_0^2 + \int_{\theta_0}^{\theta} 2\sin s ds$$

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Extension of Lyapunov lemma

Let θ_0 be an equilibrium of the system

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0,$$

that is, the solution of the equation $\gamma(\theta_0) = 0$.

Suppose $\alpha(\cdot)$ and $\gamma(\cdot)$ are C^1 at θ_0, and $\beta(\cdot)$ is continuously differentiable in some vicinity of θ_0, but not necessarily at θ_0.

Consider the second order auxiliary linear system (the linearization of nonlinear one at θ_0, if $\beta(\cdot)$ would be C^1-smooth at θ_0)

$$\frac{d^2}{dt^2}z + \left[\frac{d}{d\theta}\frac{\gamma(\theta)}{\alpha(\theta)}\right]\Bigg|_{\theta=\theta_0} \cdot z = 0$$

If the auxiliary linear system has the center at $z = 0$, then the nonlinear system has the center at the equilibrium θ_0

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Example

In the first Lecture, we have seen the system

$$\ddot{\theta} + \left[\sin\left(100|\theta|\right) - \sqrt{|\theta|}\right]\dot{\theta}^2 + a \cdot \sin\theta = 0$$

Its linearization at the equilibrium

$$\theta_0 = 0$$

does not exist and one **cannot** apply Lyapunov lemma for this system: **its coefficients are not analytic functions.**

But the statement says that, in fact, this system has the centre at

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whenever a is positive.

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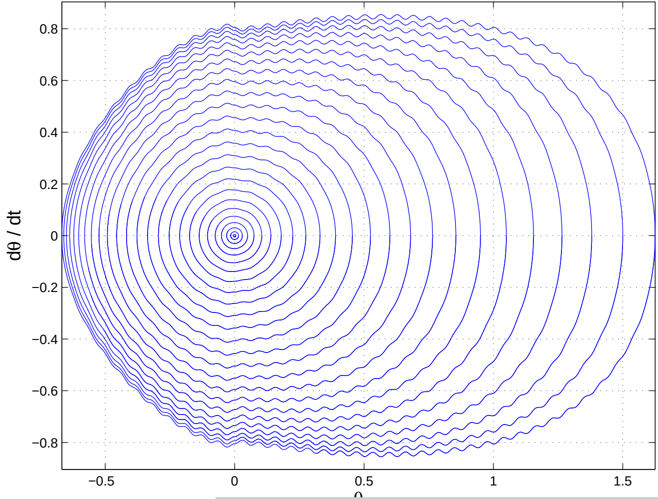

The phase portrait of $\ddot{\theta} + \left[\sin\left(100|\theta|\right) - \sqrt{|\theta|}\right]\dot{\theta}^2 + \sin\theta = 0$

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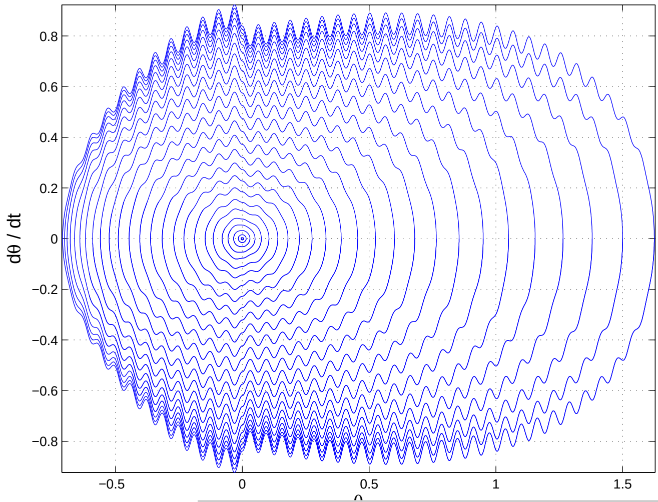

The phase portrait of $\ddot{\theta} + \left[5 \cdot \sin\left(100|\theta|\right) - \sqrt{|\theta|}\right] \dot{\theta}^2 + \sin\theta = 0$

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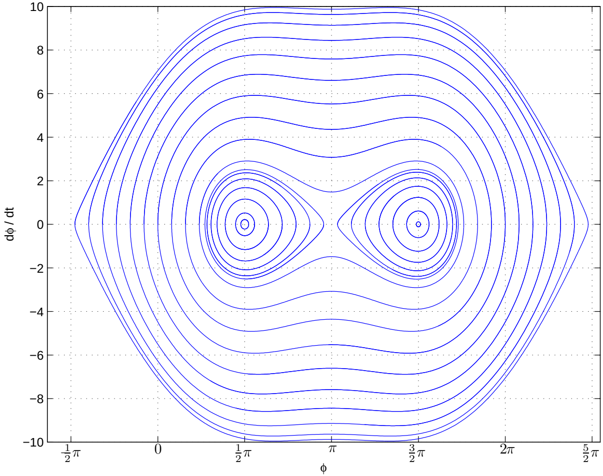


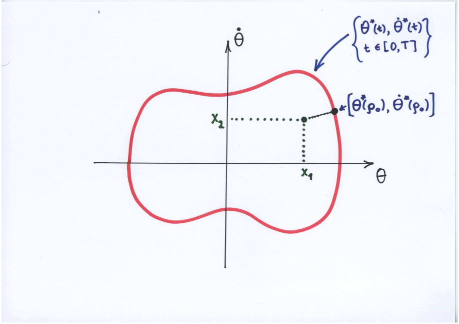
The phase portrait of a reduced system with large cycles surrounding several equilibria.

Integral is a distance

Given the target orbit $[\theta^\star(t), \dot\theta^\star(t)]$, then

- For any x_1 and x_2 the function $I(\cdot)$ satisfies the identity

$$I\left(x_1, x_2, \theta^\star(0), \dot\theta^\star(0)\right) \equiv I\left(x_1, x_2, \theta^\star(\rho), \dot\theta^\star(\rho)\right), \quad \rho \in [0, T]$$



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- Nearby the target orbit $|I(\cdot)|$ measures the distance to the orbit. Namely, the following approximation holds

$$I\left(x_1, x_2, \theta^\star(\rho_0), \dot\theta^\star(\rho_0)\right)^2 =$$

$$= \min_{0\le\rho<T}\left\{|x_1 - \theta^\star(\rho)|^2 + |x_2 - \dot\theta^\star(\rho)|^2\right\} \times$$

$$\times 4\left[\dot\theta^\star(\rho_0)^2 + \ddot\theta^\star(\rho_0)^2\right] + \ldots$$

Here

$$\rho_0 = \arg\min_{0\le\rho<T}\left\{|x_1 - \theta^\star(\rho)|^2 + |x_2 - \dot\theta^\star(\rho)|^2\right\}$$

Passivity relation for reduced dynamics

The time derivative of the function $I(\theta, \dot{\theta}, x, y)$ defined as

$$I = \dot{\theta}^2 - \exp\left\{-2\int_x^\theta \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\}\left[y^2 - \int_x^\theta \exp\left\{2\int_x^s \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\} \frac{2\gamma(s)}{\alpha(s)} ds\right]$$

with x and y being some constants, calculated along a solution of the system

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = W$$

has the form

$$\frac{d}{dt} I(\theta, \dot{\theta}, x, y) = \dot{\theta}\left\{\frac{2}{\alpha(\theta)} W - \frac{2\beta(\theta)}{\alpha(\theta)} I(\theta, \dot{\theta}, x, y)\right\}$$

Example: a cart-pendulum system

Planning oscillations of a cart-pendulum system

A searched oscillation of the pendulum around the upright equilibrium for the system

$$\begin{aligned} 2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 &= u \\ \cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta &= 0 \end{aligned}$$

can be written through the nested representation as

- $\theta^*(\cdot)$ defined a as solution of the reduced dynamics

$$\left[1 + \cos\theta \cdot \phi'(\theta)\right] \ddot{\theta} + \cos\theta \cdot \phi''(\theta) \cdot \dot{\theta}^2 - g \cdot \sin\theta = 0$$

- $x^*(\cdot)$ defined by the kinematic relation

$$x^*(t) = \phi\left(\theta^*(t)\right)$$

Planning oscillations of a cart-pendulum system

A searched oscillation of the pendulum around the upright equilibrium for the system

$$2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 = u$$

$$\cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta = 0$$

can be written through the nested representation as

- $\theta^*(\cdot)$ defined a as solution of the reduced dynamics

$$\Big[1 + \cos\theta \cdot \phi'(\theta)\Big]\,\ddot{\theta} + \cos\theta \cdot \phi''(\theta) \cdot \dot{\theta}^2 - g \cdot \sin\theta = 0$$

- $x^*(\cdot)$ defined by the kinematic relation

$$x^*(t) = \phi\left(\theta^*(t)\right) + v_x \cdot t$$

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The system

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has the center at $\theta_e = 0$, if the following linear system

$$\ddot{z} + \left.\left[\frac{d}{d\theta}\left(\frac{-g \cdot \sin\theta}{1 + \cos\theta \cdot \phi'(\theta)}\right)\right]\right|_{\theta=0} \cdot z = 0$$

has the center at the equilibrium $z = 0$

The calculations show that the linearization has the form

$$\ddot{z} + \omega^2 z = 0$$

if

$$\frac{-g}{1 + \phi'(0)} = \omega^2 \qquad \left(\Rightarrow \quad \phi'(0) = -\left[1 + \frac{g}{\omega^2}\right]\right)$$

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Consider the following function

$$x = \phi(\theta) = -\left[1 + \frac{g}{\omega^2}\right] \ln\left(\frac{1 + \sin\theta}{\cos\theta}\right)$$

where ω is the desired frequency of oscillations.

One can check that for this particular choice

$$\phi'(\theta) = -\left[1 + \frac{g}{\omega^2}\right] \frac{1}{\cos\theta} \quad \Rightarrow \quad \phi'(0) = -\left[1 + \frac{g}{\omega^2}\right]$$

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The dynamics of the cart-pendulum system

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projected on the **synchronization pattern**

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$$-\frac{g}{\omega^2}\ddot{\theta} - \left[1 + \frac{g}{\omega^2}\right]\frac{\sin\theta}{\cos\theta}\dot{\theta}^2 - g \cdot \sin\theta = 0$$

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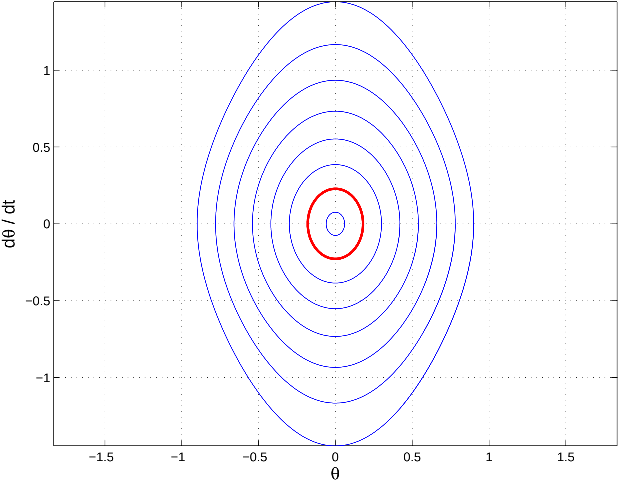
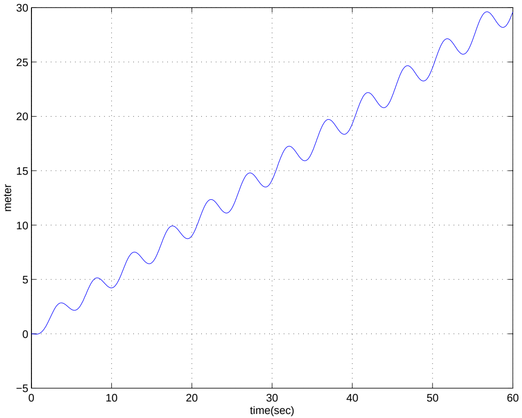


The phase portrait of the reduced dynamics when $\omega = 2\pi/5$ [rad/s]



The position of the cart for the chosen $\theta^*(t)$ behavior with

$$x^*(t) = -\left[1 + \frac{g}{\omega^2}\right] \ln\left(\frac{1 + \sin\theta^*(t)}{\cos\theta^*(t)}\right) + 0.5 \cdot t$$