Lecture 6

Orbital Stability and Stabilization for Underactuated Systems

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UNAM, Mexico
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Learning outcomes: Concepts of moving Ponicare sections, transverse coordinates and transverse linearization for a solution of a nonlinear system. Andronov-Vitt theorem. Choices of transverse coordinates and their linarization for motions of underactuated mechanical systems. Examples

Outline

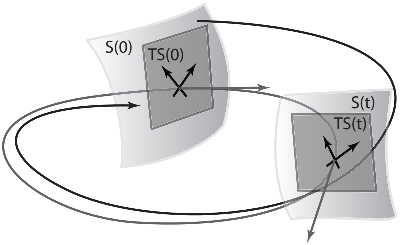
Transverse dynamics

Dynamics in a vicinity of a cycle

Given a T-periodic solution $x^*(\cdot)$,
$x^*(t) = x^*(t+T)\ \forall\, t$, of the system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^{2n},$$

for analyzing its local properties introduce a family of transverse sections $\{S(\tau)\}$, $\tau \in [0, T]$, which union can recreate a tubing vicinity of the cycle.



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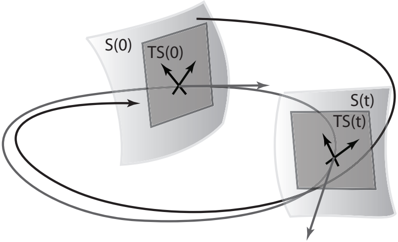

Defining hypersurfaces $\{S(\tau)\}_{\tau\in[0,T]}$ implies a change of coordinates:

$$\mathbb{R}^{2n} \ni x(\tau) \mapsto \left[\theta(\tau) \in \mathbb{R}^1;\, x_\perp \in S(\tau)\right], \qquad \tau \in [0, T]$$

such that in new coordinates the periodic solution $x^*(\cdot)$ is of the form

$$\theta = \theta^*(t), \quad x_\perp^*(t) \equiv \mathbf{0}, \qquad \forall\, t$$

Orbital asymptotic stability of $x^*(\cdot)$ means that $x_\perp(t) \to 0$ as $t \to \infty$.

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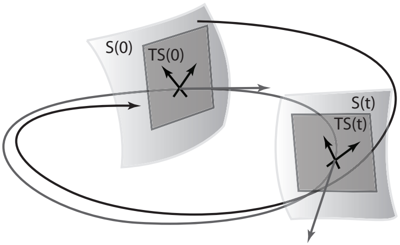

The linearization of the full dynamics of the system in a vicinity of $x^*(\cdot)$

$$\dot{z} = \left[\frac{\partial}{\partial x} f(x)\right]\Big|_{x=x^*(t)} z = A(t)z, \quad z \in \mathbb{R}^{2n}$$

- cannot be asymptotically stable since $\theta(t) \not\to 0$ as $t \to +\infty$

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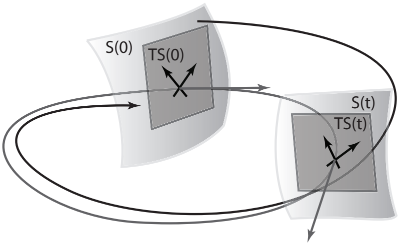

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- cannot be asymptotically stable since $\theta(t) \not\to 0$ as $t \to +\infty$
- has a linear invariant subspace of co-dimension 1, which vectors approximate time evolution of transverse coordinates $x_\perp(\cdot)$.

Andronov-Vitt theorem (1930)

Given a T-periodic solution

$$x^*(t) = x^*(t+T) \quad \forall\, t$$

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Consider the $(2n \times 2n)$-matrix function $\Phi(\cdot)$ defined as a solution of

$$\tfrac{d}{dt}\Phi(t) = A(t)\Phi(t), \qquad \Phi(0) = I_{2n}$$

One of eigenvalues of the monodromy matrix $\Phi(T)$ equals to 1.

If the amplitudes of $(2n-1)$ eigenvalues of the matrix $\Phi(T)$ are less than 1, then the solution $x^*(\cdot)$ is exponentially orbitally stable.

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Challenges in orbital feedback stabilization

Given a T-periodic solution $x^*(\cdot)$ of the nonlinear control system

$$\dot{x} = f(x) + g(x)u, \qquad x \in \mathbb{R}^{2n},\ u \in \mathbb{R}^m,$$

obtained in response of an input $u^*(\cdot) \equiv 0$, consider the task to design a controller for asymptotic orbital stabilization of $x^*(\cdot)$

Suppose it exists, then such orbitally stabilizing feedback controller

$$u = U(x)$$

will satisfy the interpolation condition

$$U(x)\big|_{x=x^*(t)} = u^*(t) \equiv 0, \quad \forall\, t$$

For a smooth $U(\cdot)$, it can be written as (see the Hadamard lemma)

$$u = K(x)x_\perp(x), \qquad x_\perp \in \mathbb{R}^{2n-1}$$

where $x_\perp(\cdot)$ are transverse coordinates defined for the motion $x^*(\cdot)$

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The system dynamics in a vicinity of the solution $x^*(\cdot)$ can be written in new coordinates

$$\tilde{x} = [\theta;\, x_\perp], \quad \theta \in \mathbb{R}^1,\ x_\perp \in \mathbb{R}^{2n-1}$$

as

$$\frac{d}{dt}\tilde{x} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})u.$$

With a feedback controller candidate

$$u = K(\tilde{x})x_\perp$$

the closed loop system becomes

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The linearization of the closed loop system in a vicinity of $x^*(\cdot)$ is

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By Andronov-Vitt theorem the linear feedback controller with the gain $k(\cdot)$ is not stabilizing the origin of this system

Challenges in orbital feedback stabilization

Orbital stabilization of the solution $x^*(\cdot)$ the system **by linearization**

$$x \in \mathbb{R}^{2n} : \quad \frac{d}{dt}x = f(x) + g(x)u, \quad [x^*(\cdot);\ u^*(\cdot)]$$

requires stabilization of **linearization** of transverse coordinates dynamics

This means that with change of coordinates

$$x \mapsto \tilde{x} = [\theta \in \mathbb{R}^1;\ x_\perp \in \mathbb{R}^{2n-1}]$$

it requires stabilizing an **invariant subspace** of the linearization

$$\dot{z} = A(t)z + B(t)v, \quad v = k(t)\delta x_\perp, \quad z \in \mathbb{R}^{2n},\ \delta x_\perp \in \mathbb{R}^{2n-1},\ v \in \mathbb{R}^n$$

of the system defined by variations of transverse coordinates $x_\perp(\cdot)$:

$$\delta x_{1\perp} \equiv 0, \quad \delta x_{2\perp} \equiv 0, \quad \ldots, \quad \delta x_{(2n-1)\perp} \equiv 0$$

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Generic choice of transverse coordinates

Given a T-periodic pair $[\, x^*(\cdot), u^*(\cdot) \,]$ of the system

$$\frac{d}{dt}x = f(x) + g(x)u, \quad x \in \mathbb{R}^{2n},\ u \in \mathbb{R}^m$$

how to define the transverse coordinates $x_\perp \in \mathbb{R}^{2n-1}$ for the motion?

Step 1: Given $t_\bullet \in [0, T]$ and the vector $v_0 := \frac{d}{dt} x^*(t_\bullet)$, choose a set of vectors

$$v_1 \in \mathbb{R}^{2n}, \quad v_2 \in \mathbb{R}^{2n}, \quad \ldots, \quad v_{2n-1} \in \mathbb{R}^{2n}$$

such that the $(2n \times 2n)$-matrix

$$[v_0,\, v_1,\, v_2,\, \ldots,\, v_{2n-1}]$$

is of rank $2n$

Step 2: Choose $v_i(t)$ for all $t \in [0, T]$ to be periodic and smooth.

Generic choice of transverse coordinates

Given a T-periodic pair $[\, x^*(\cdot), u^*(\cdot)\,]$ of the system

$$\tfrac{d}{dt}x = f(x) + g(x)u, \quad x \in \mathbb{R}^{2n},\ u \in \mathbb{R}^m$$

how to define the transverse coordinates $x_\perp \in \mathbb{R}^{2n-1}$ for the motion?

Step 1: Given $t_\bullet \in [0, T]$ and the vector $v_0 := \frac{d}{dt} x^*(t_\bullet)$, choose a set of vectors

$$v_1 \in \mathbb{R}^{2n}, \quad v_2 \in \mathbb{R}^{2n}, \quad \ldots, \quad v_{2n-1} \in \mathbb{R}^{2n}$$

such that the $(2n \times 2n)$-matrix

$$[v_0,\, v_1,\, v_2,\, \ldots,\, v_{2n-1}]$$

is of rank $2n$

Step 2: Choose $v_i(t)$ for all $t \in [0, T]$ to be periodic and smooth.

Generic choice of transverse coordinates

Given a T-periodic pair $[\,x^*(\cdot), u^*(\cdot)\,]$ of the system

$$\frac{d}{dt}x = f(x) + g(x)u, \quad x \in \mathbb{R}^{2n},\ u \in \mathbb{R}^m$$

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Transverse coordinates

Alternative Representations of a Motion

Given a motion $q^*(t)$, $t \in [0, T]$ of a mechanical underactuated system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = B(q)u, \qquad q \in \mathbb{R}^n,\ u \in \mathbb{R}^{n-1}$$

obtained in response to the input signal $u^*(t)$, $t \in [0, T]$

---

if the motion admits kinematic relations

$$q_1 = \phi_1(\theta), \quad q_2 = \phi_2(\theta), \quad \ldots, \quad q_n = \phi_n(\theta)$$

valid between the coordinates q_i and a new scalar variable θ

---

then there is a scalar system with coefficients defined by $\mathcal{L}$ and $\{\phi_i\}_{i=1}^n$

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0 \qquad \text{and its solution } \theta^*(\cdot)$$

---

which allows re-writing the motion $q^*(\cdot)$ on the time interval $[0, T]$ as

$$q_1^*(t) = \phi_1(\theta^*(t)),\ q_2^*(t) = \phi_2(\theta^*(t)),\ \ldots,\ q_n^*(t) = \phi_n(\theta^*(t))$$

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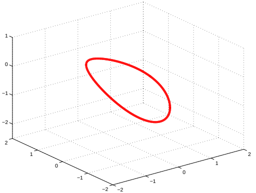
Transverse Coordinates for a Motion :



Given a T-periodic motion of n-DOF MS

$$q^\star(t) = \Big[q_1^\star(t);\, q_2^\star(t);\, \ldots;\, q_n^\star(t)\Big] = q^\star(t+T),$$

a scalar θ, the functions for its representation

$$\phi_1(\cdot), \quad \phi_2(\cdot), \quad \ldots, \quad \phi_n(\cdot)$$

The $(n+1)$ coordinates for describing the dynamics in a vicinity of $q^\star(t)$

$$\theta, \quad y_1 := q_1 - \phi_1(\theta), \quad \ldots, \quad y_n := q_n - \phi_n(\theta)$$

are excessive! And, therefore, they are not independent and one variable is redundant!

Then one of candidates for transverse coordinates $x_\perp$ for a motion of the system with $x = (q_1, \ldots, q_n, \dot q_1, \ldots, \dot q_n)^T$ is given by $(2n-1)$-quantities

$$x_\perp = \Big[I(\theta, \dot\theta, \theta^\star(0), \dot\theta^\star(0));\, y_1;\, \ldots;\, y_{n-1};\, \dot y_1;\, \ldots;\, \dot y_{n-1}\Big]$$

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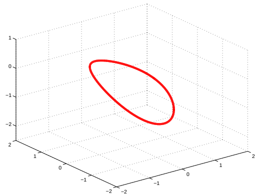

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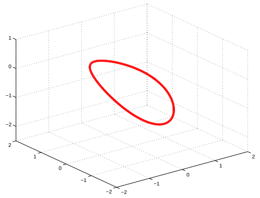

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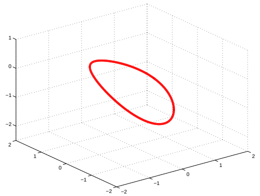

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Transverse linearization

Steps in Computing Transverse Linearization

Given a T-periodic motion $q^\star(t) = \left(q_1^\star(t),\, q_2^\star(t),\, \ldots,\, q_n^\star(t)\right)^T$ of the mechanical system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = B(q)u$$

Step 1: Rewrite the dynamics in the coordinates: θ, y with

$$y_1 = q_1 - \phi_1(\theta), \quad \ldots, \quad y_{n-1} = q_{n-1} - \phi_{n-1}(\theta)$$

Step 2: Define a feedback transformation

$$u = (u_1, \ldots, u_{n-1}) \to v = (v_1, \ldots, v_{n-1})$$

such that the dynamics of y become linear

$$\ddot{y} = v, \qquad \ddot{\theta} = N(\theta, \dot{\theta}, y, \dot{y}, \ddot{y})$$

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Steps in Computing Transverse Linearization: θ-dynamics

Since the θ-dynamics with constraints $y = \dot{y} = \ddot{y} = v = 0$

$$\ddot{\theta} = N(\theta, \dot{\theta}, y, \dot{y}, \ddot{y})$$

become

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

Therefore, the dynamics of θ-variable can be rewritten as

$$\begin{aligned}\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) \;\; = \;\; & g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\, y + \\ & + g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\, \dot{y} + g_v(\theta, \dot{\theta}, y, \dot{y})\, \ddot{y}\end{aligned}$$

The functions $g_y(\cdot)$, $g_{\dot{y}}(\cdot)$ and $g_v(\cdot)$ are to be found based on

- the Lagrangian $\mathcal{L}(\cdot)$ of the system;
- the functions $\phi_1(\cdot), \ldots, \phi_{n-1}(\cdot)$ representing the motion

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Computing Transverse Linearization: Passivity Relation for $I(\cdot)$

The mechanical system written in y, θ-coordinates is

$$\begin{aligned}
\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) &= g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})y + \\
&\quad + g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\dot{y} + g_v(\theta, \dot{\theta}, y, \dot{y})v \\
\ddot{y} &= v
\end{aligned}$$

We are searching for linearization of transverse dynamics

$$x_{\perp} = \left[\, I(\theta, \dot{\theta}, \theta^{\star}(0), \dot{\theta}^{\star}(0));\ y;\ \dot{y} \,\right]$$

around the target solution: $\theta(t) = \theta^{\star}(t)$, $y(t) \equiv 0$

To do so, one can use the passivity relation for $\frac{d}{dt}I(\cdot)$ found previously

$$\begin{aligned}
\frac{d}{dt}I(\theta, \dot{\theta}, a, b) &\equiv \dot{\theta}\left\{\frac{2}{\alpha(\theta)}W - \frac{2\beta(\theta)}{\alpha(\theta)}I(\theta, \dot{\theta}, a, b)\right\} \\
&\equiv \dot{\theta}\left\{\frac{2}{\alpha(\theta)}\left[g_y(\cdot)y + g_{\dot{y}}(\cdot)\dot{y} + g_v(\cdot)v\right] - \frac{2\beta(\theta)}{\alpha(\theta)}I(\theta, \dot{\theta}, a, b)\right\}
\end{aligned}$$

Computing Transverse Linearization: Passivity Relation for $I(\cdot)$

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$$\begin{aligned}\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) &= g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})y + \\ &\quad + g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\dot{y} + g_v(\theta, \dot{\theta}, y, \dot{y})v \\ \ddot{y} &= v\end{aligned}$$

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Transverse Linearization: the Final Form

The linear control system with $(2n-1)$ states and $(n-1)$ control inputs

$$\begin{aligned} \frac{d}{dt} I_{\heartsuit}(t) &= \frac{2\,\dot{\theta}^{\star}(t)}{\alpha(\theta^{\star}(t))}\Big\{\Big[\tilde{g}_y(t) y_{\heartsuit}(t) + \tilde{g}_{\dot{y}}(t)\dot{y}_{\heartsuit}(t) + \tilde{g}_v(t) v_{\heartsuit}(t)\Big] - \\ &\qquad - \beta(\theta^{\star}(t)) I_{\heartsuit}(t)\Big\} \\ \ddot{y}_{\heartsuit}(t) &= v_{\heartsuit}(t) \end{aligned}$$

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with the functions

$$\begin{aligned}\tilde g_y(t) &= g_y(\theta,\dot\theta,\ddot\theta,y,\dot y)\Big|_{\substack{\theta=\theta^\star(t),\ \dot\theta=\dot\theta^\star(t)\\ \ddot\theta=\ddot\theta^\star(t),\ y=\dot y=0}} \\ \tilde g_{\dot y}(t) &= g_{\dot y}(\theta,\dot\theta,\ddot\theta,y,\dot y)\Big|_{\substack{\theta=\theta^\star(t),\ \dot\theta=\dot\theta^\star(t)\\ \ddot\theta=\ddot\theta^\star(t),\ y=\dot y=0}} \\ \tilde g_v(t) &= g_v(\theta,\dot\theta,y,\dot y)\Big|_{\substack{\theta=\theta^\star(t),\ \dot\theta=\dot\theta^\star(t)\\ y=\dot y=0}}\end{aligned}$$

Transverse Linearization of the Mechanical System in a Vicinity of the Motion $q^\star(\cdot)$

Transverse Linearization: the Final Form

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Any **linear** feedback controller of the form

$$v_{\heartsuit}(t) = K(t) \begin{bmatrix} I_{\heartsuit} \\ y_{\heartsuit} \\ \dot{y}_{\heartsuit} \end{bmatrix}, \quad K(t) = K(t+T),$$

that stabilizes the origin of the linear control system can be transformed into a **nonlinear controller that stabilizes orbitally** the nominal periodic motion $q^{\star}(\cdot)$ of the nonlinear underactuated mechanical system