

Lecture 7

Orbital Stability and Stabilization for Hybrid Systems

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Learning outcomes: Hybrid dynamics and motion planning. Hybrid transverse linearization for a solution of a mechanical system. Examples

1. Problem formulation and examples
 - Gaits of a passive compass biped
 - Perpetual rotations of a devil stick
2. Planning a cycle of a hybrid mechanical system
3. Orbital stabilization of a hybrid cycle
 - Choices for moving Poincare sections
 - Hybrid transverse linearization
4. Analysis of gaits of a passive compass biped

Problem formulation

Problem formulation

Given

- Controlled mechanical system

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = B(q)u,$$

with $q = [q_1, \dots, q_n]$, $u = [u_1, \dots, u_m]$

Tasks:

- To plan a hybrid motion, i.e. a pair $[q_*(\cdot), u_*(\cdot)]$
- To design a controller to achieve contraction to the orbit of the motion $q_*(\cdot)$

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- Discrete dynamics (instantaneous maps)

$$F^{(i)} : \Gamma_-^{(i)} \rightarrow \Gamma_+^{(i)} \quad i = 1, \dots, N_d$$

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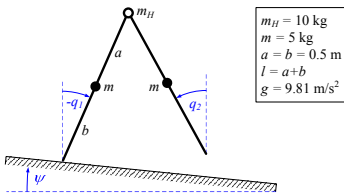
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Gaits of a passive compass biped



The continuous-in-time arc of the hybrid gait is the solution of Euler-Lagrange equations shaped by a zero control input:

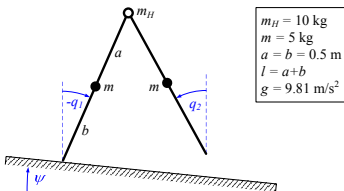
$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_1} \right] - \frac{\partial \mathcal{L}}{\partial q_1} = 0,$$

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_2} \right] - \frac{\partial \mathcal{L}}{\partial q_2} = 0,$$

where the Lagrangian is

$$\mathcal{L} = \frac{1}{2} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}^T \underbrace{\begin{bmatrix} p_1 & -p_2 \cos(q_1 - q_2) \\ -p_2 \cos(q_1 - q_2) & p_3 \end{bmatrix}}_{= M(q)} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} - \underbrace{[p_4 (\cos(q_1) - 1) + p_5 (1 - \cos(q_2))]}_{= V(q)}$$

Gaits of a passive compass biped



The discrete dynamics appear due to impact activated when the continuous-in-time solution hits the surface

$$\Gamma_- = \{[q, \dot{q}] : \cos(q_1 + \psi) - \cos(q_2 + \psi) = 0\}$$

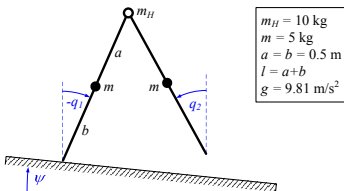
The discrete dynamics change the state instantaneously by the rule

$$F : \Gamma_- \ni [q_-, \dot{q}_-] \mapsto [q_+, \dot{q}_+] \in \Gamma_+$$

where

$$q_+ = F_1 q_-, \quad \dot{q}_+ = F_2(q_-) \dot{q}_- \quad \Gamma_+ = \Gamma_-$$

Gaits of a passive compass biped



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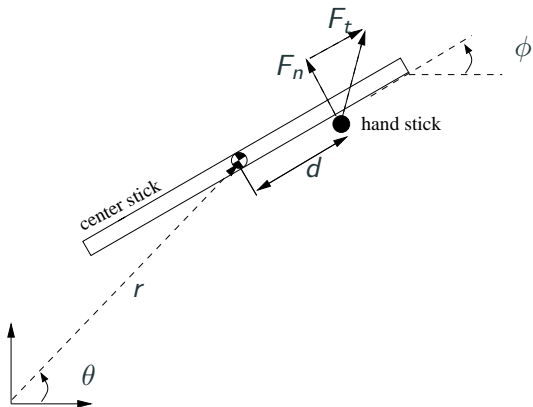
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The task is to find a periodic gait and analyze its stability

Perpetual rotations of a devil stick



Devil stick consists of a (rotating) **centre stick** driven by a force generated by a **hand stick**, which in turn rolls on the boundary of centre stick without slipping

Perpetual rotations of a devil stick

The dynamics of the centre stick written in polar coordinates are

$$\begin{aligned}\ddot{r} &= r\dot{\theta}^2 - g \sin \theta + \frac{\cos(\theta - \phi)}{m} F_t + \frac{\sin(\theta - \phi)}{m} F_n \\ \ddot{\theta} &= -\frac{2\dot{r}\dot{\theta}}{r} - \frac{g \cos \theta}{r} - \frac{\sin(\theta - \phi)}{rm} F_t + \frac{\cos(\theta - \phi)}{rm} F_n \\ \ddot{\phi} &= \frac{d(\phi)}{J} F_n = \frac{-\rho\phi + d_0}{J} F_n\end{aligned}$$

- (r, θ) are coordinates for center of mass of centre stick;
- ϕ is the angle that center stick makes with horizontal;
- $d(\phi)$ is the instantaneous position at which the centre stick and the hand stick are in contact $d(\phi) = -\rho\phi + d_0$;
- d_0 is the initial contact position when $\phi = 0$;
- ρ is the radius of the hand stick, m is the mass of the centre stick, J is its moment of inertia;
- F_t and F_n are tangential and normal components of force.

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Searching for a propeller motion for the centre stick

The task are to **create** a propeller motion by $F_t(\cdot)$, $F_n(\cdot)$

- being “periodic” in coordinates

$$\phi(T) = \phi(0) + 2\pi, \quad \theta(T) = \theta(0) + 2\pi, \quad r(T) = r(0)$$

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$$\lim_{\varepsilon \rightarrow 0_+} \frac{d}{dt}\phi(T - \varepsilon) = \lim_{\varepsilon \rightarrow 0_+} \frac{d}{dt}\phi(T + \varepsilon)$$

even at $t = T$, where the hand and centre sticks change instantaneously the **contact point** in-between to its original value at $t = 0$.

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and to **stabilize** the hybrid cycle.

For both examples, the discrete dynamics are necessary for the description of the nominal motion

But the reasons the update of the state included in the description are different

- For the compass biped the update is due to the physics of the impact forces

Origin of discrete dynamics

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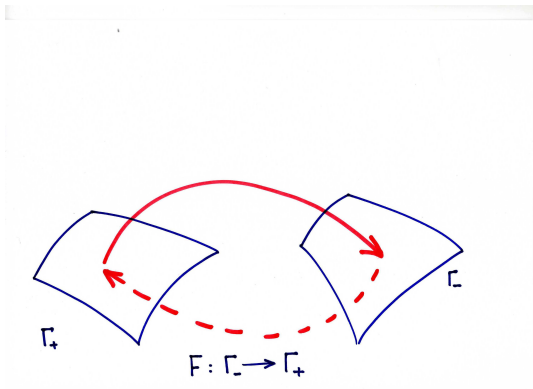
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But the reasons the update of the state included in the description are different

- For the compass biped the update is due to the physics of the impact forces
- For the devil stick, the update is due to the control strategy
- In other examples, mechanical systems can instantly change a number of degrees of freedom!

Planning a hybrid cycle

Hybrid cycle with one jump



A cycle consists of a sub-arc of the continuous-in-time motion and the update map, which brings the end point to the point where the cycle starts

Hybrid cycle with one jump

Assume that a motion of a mechanical system with n degrees of freedom

$$q_1 = q_1(t), \quad q_2 = q_2(t), \quad \dots, \quad q_n = q_n(t), \quad t \in [0, T]$$

can be represented as

$$q_1(t) = \phi_1(\theta(t)), \quad q_2(t) = \phi_2(\theta(t)), \quad \dots, \quad q_n(t) = \phi_n(\theta(t)),$$

where the scalar variable

$$\theta = \theta(t), \quad t \in [0, T]$$

is a generator of the motion.

If the system is underactuated then $\theta(\cdot)$ is a solution of

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

Hybrid cycle with one jump

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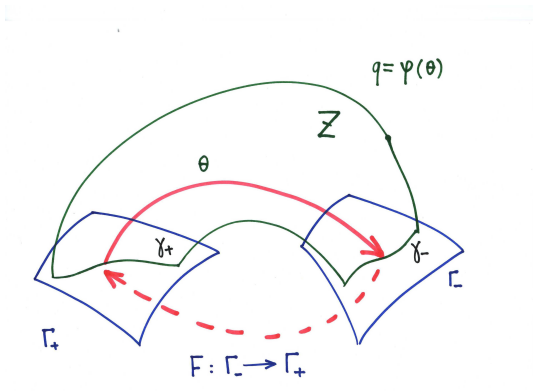
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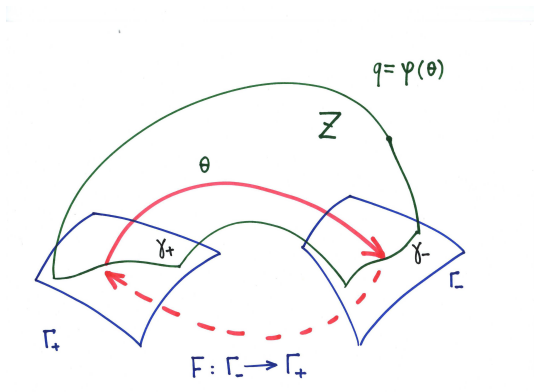
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Hybrid cycle with one jump



The relations $q = \phi(\theta)$ associated with the motion define a 2-D surface Z in the state space of a robot. The curves γ_+ and γ_- are the intersections of Z with Γ_+ and Γ_- respectively.

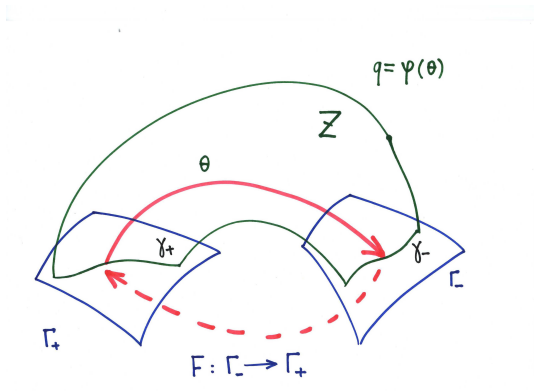
Hybrid cycle with one jump



Behavior of $\theta(\cdot)$ along the motion is not any, it is the solution of

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

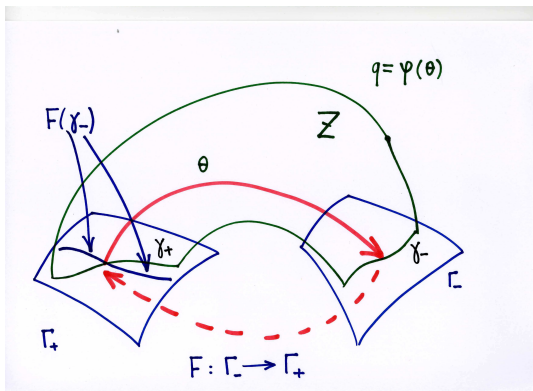
Hybrid cycle with one jump



Integrability $\Rightarrow [\theta(T_h), \dot{\theta}(T_h)]$ can be found by solving Eqns:

$$I([\theta(T_h), \dot{\theta}(T_h)], [\theta(0), \dot{\theta}(0)]) = 0, \quad [q(\theta(T_h)), \dot{q}(\theta(T_h))] \in \Gamma_-$$

Hybrid cycle with one jump



The discrete dynamics of the hybrid mechanical system **do not** necessary keep the 2-D manifold Z invariant. In general,

$$F(\gamma_-) \notin \gamma_+$$

Hybrid cycle with one jump

- Choose a parametric set of C^2 -smooth functions

$$\phi(\theta, P) = \{\phi_1(\theta, P), \phi_2(\theta, P), \dots, \phi_n(\theta, P)\}$$

Hybrid cycle with one jump

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- Compute 2-D manifolds $Z(P)$ and projected dynamics

$$\alpha(\theta, P)\ddot{\theta} + \beta(\theta, P)\dot{\theta}^2 + \gamma(\theta, P) = 0$$

Any solution $\theta(t, P)$, if exists till $t = \tau_1 > 0$, satisfies

$$I([\theta(\tau_1, P), \dot{\theta}(\tau_1, P)], [\theta(0, P), \dot{\theta}(0, P)]) = 0$$

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- Compute curves: $\gamma_+ = \Gamma_+ \cap Z(P)$, $\gamma_- = \Gamma_- \cap Z(P)$ and

$$\mathcal{F}([\theta, \dot{\theta}])\Big|_{[\theta, \dot{\theta}] \in \gamma_-} = F([q, \dot{q}])\Big|_{\substack{q = \phi(\theta, P) \\ \dot{q} = \phi'(\theta, P) \dot{\theta}}}$$

Hybrid cycle with one jump

- $\phi(\theta, P) = \{\phi_1(\theta, P), \phi_2(\theta, P), \dots, \phi_n(\theta, P)\}$
- Compute 2-D manifolds $Z(P)$ and the integral

$$I\left([\theta(\cdot, P), \dot{\theta}(\cdot, P)], [\theta(0, P), \dot{\theta}(0, P)]\right) = 0$$

- Compute $\gamma_+(P)$, $\gamma_-(P)$ and $\mathcal{F}([\theta, \dot{\theta}], P)$

Hybrid cycle with one jump

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Organize a search for

parameters $P = P_*$ and the scalars a, b, x, y

that solve the equations

$$I([a, b], [x, y], P) = 0$$

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Hybrid cycle with one jump

Consider a system that lives in $[\theta, \dot{\theta}]$ -plane with $\theta \geq 0$: its continuous dynamics are defined for $\theta > 0$ as

$$\ddot{\theta} + \theta = 0$$

and its instantaneous law is defined on $\theta = 0$ as

$$\theta_+ = 0, \quad \dot{\theta}_+ = (\dot{\theta}_-) \cdot \sin(\dot{\theta}_-)$$

To find periodic motions for this hybrid system, one needs to solve the algebraic equations $\theta_+ = 0$, $\theta_- = 0$ and

$$\begin{aligned}(\dot{\theta}_+)^2 + (\theta_+)^2 - (\dot{\theta}_-)^2 - (\theta_-)^2 &= 0 \\ \dot{\theta}_+ &= (\dot{\theta}_-) \cdot \sin(\dot{\theta}_-)\end{aligned}$$

Solutions are

$$\theta_+ = 0, \quad \dot{\theta}_+ = 2\pi \cdot n + \frac{\pi}{2}, \quad n = 0, 1, \dots$$

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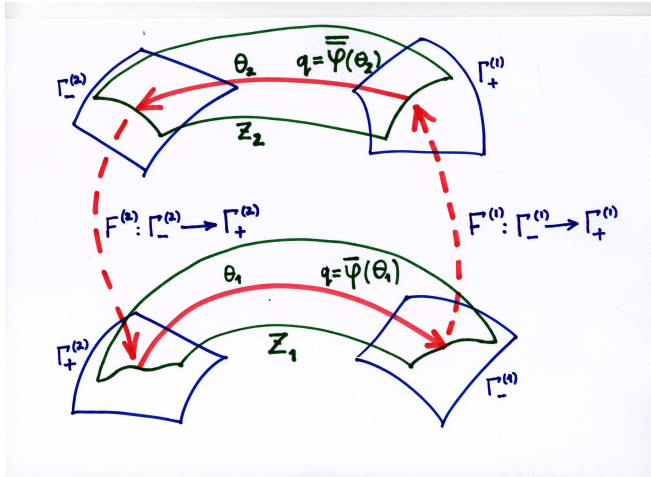
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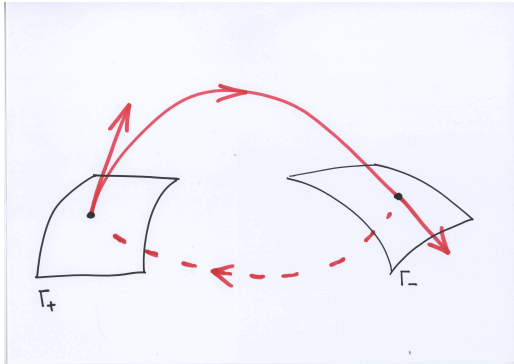
Hybrid cycle with two jumps



Similar constructions for a hybrid cycle with two jumps

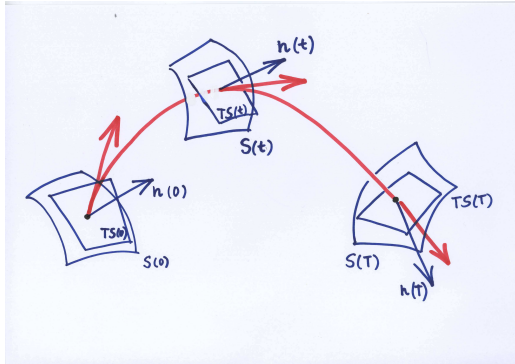
Orbital stabilization of a hybrid cycle

Hybrid cycle with one jump



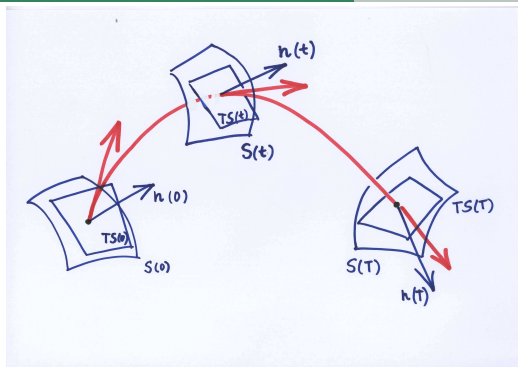
A hybrid cycle with one jump

Hybrid cycle with one jump



- **Moving Poincaré section** is a collection of $(2n - 1)$ -D surfaces $S(t)$ transversal to the continuous-in-time sub-arc;

Hybrid cycle with one jump



- **Moving Poincaré section** is a collection of $(2n - 1)$ -D surfaces $S(t)$ transversal to the continuous-in-time sub-arc;
- The state $[q, \dot{q}] \in \mathbb{R}^n \times \mathbb{R}^n$ can be transformed into $[\theta, x_{\perp}] \in \mathbb{R}^1 \times \mathbb{R}^{2n-1}$. Here $\theta(t)$ defines point along the orbit, $x_{\perp}(t) \in S(t)$ are **transverse coordinates**.

Transverse Coordinates for MS:

Given a T -periodic motion

$$q^*(t) = \left(q_1^*(t), q_2^*(t), \dots, q_n^*(t) \right)^T$$

There are a motion generator θ ; n -functions for its nested representation

$$\phi_1(\cdot), \phi_2(\cdot), \dots, \phi_n(\cdot)$$

We can always assume that one of them, i.e. $\phi_n(\cdot)$, is trivial!

New n -generalized coordinates are θ and $y = \left(y_1, \dots, y_{n-1} \right)$

$$y_1 = q_1 - \phi_1(\theta), \quad \dots, \quad y_{n-1} = q_{n-1} - \phi_{n-1}(\theta)$$

For a motion of the E-L system with $x = \left(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n \right)^T$ a candidate for transverse coordinates x_{\perp} is given by

$$x_{\perp} = \left[I(\theta, \dot{\theta}, \theta^*(0), \dot{\theta}^*(0)), y_1, \dots, y_{n-1}, \dot{y}_1, \dots, \dot{y}_{n-1} \right]^T$$

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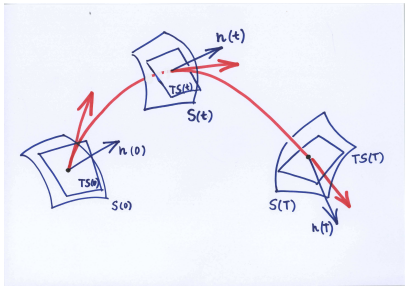
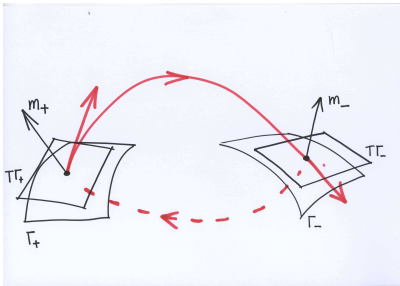
New n -generalized coordinates are θ and $y = \left(y_1, \dots, y_{n-1} \right)$

$$y_1 = q_1 - \phi_1(\theta), \quad \dots, \quad y_{n-1} = q_{n-1} - \phi_{n-1}(\theta)$$

For a motion of the E-L system with $x = \left(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n \right)^T$ a candidate for transverse coordinates x_{\perp} is given by

$$x_{\perp} = \left[I(\theta, \dot{\theta}, \theta^*(0), \dot{\theta}^*(0)), y_1, \dots, y_{n-1}, \dot{y}_1, \dots, \dot{y}_{n-1} \right]^T$$

Hybrid transverse linearization

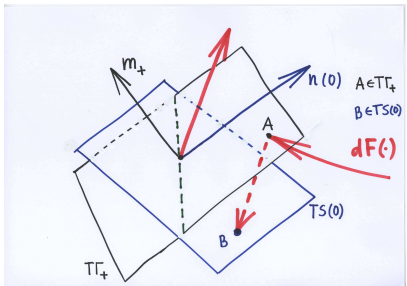
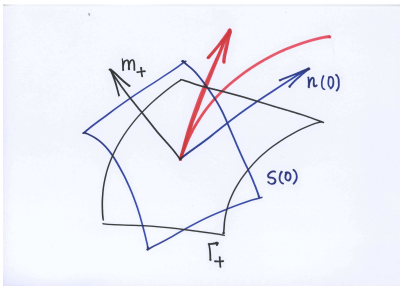
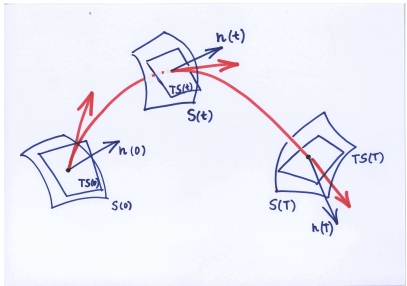
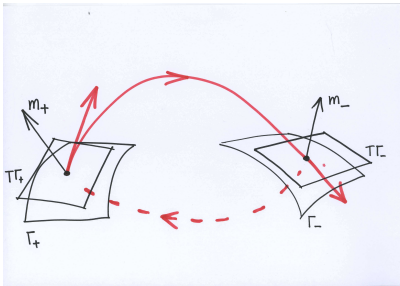


Two linear mappings

- Linearization of the update law $F(\cdot)$
- Linearization of the dynamics of transverse coordinates x_{\perp}

should be combined!

Hybrid transverse linearization



Hybrid transverse linearization

Let $q_*(t) = q_*(t + T)$, $u_*(t) = u_*(t + T)$ be a hybrid cycle of

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = B(q, \dot{q})u, \quad F : \Gamma_- \rightarrow \Gamma_+$$

Introduce a linear hybrid system defined by

- linear continuous time system

$$\frac{d}{d\tau}z = A(\tau \bmod T)z + B(\tau \bmod T)v$$

activated on time intervals

$$[0, T], [T, 2T], \dots, [kT, (k+1)T], \dots$$

It is responsible for orbital stability/stabilization of a cycle

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that re-initializes the state at times $T, 2T, \dots, kT, \dots$

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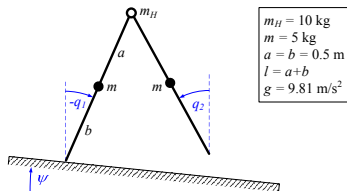
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Analysis of gaits of a passive compass biped

Gaits of a passive compass biped



The continuous-in-time arc of the hybrid gait is the solution of Euler-Lagrange equations shaped by a zero control input:

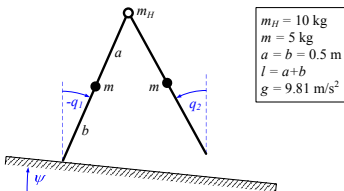
$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_1} \right] - \frac{\partial \mathcal{L}}{\partial q_1} = 0,$$

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_2} \right] - \frac{\partial \mathcal{L}}{\partial q_2} = 0,$$

where the Lagrangian is

$$\mathcal{L} = \frac{1}{2} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}^T \underbrace{\begin{bmatrix} p_1 & -p_2 \cos(q_1 - q_2) \\ -p_2 \cos(q_1 - q_2) & p_3 \end{bmatrix}}_{= M(q)} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} - \underbrace{[p_4 (\cos(q_1) - 1) + p_5 (1 - \cos(q_2))]}_{= V(q)}$$

Gaits of a passive compass biped



The discrete dynamics appear due to impact activated when the continuous-in-time solution hits the surface

$$\Gamma_- = \{[q, \dot{q}] : \cos(q_1 + \psi) - \cos(q_2 + \psi) = 0\}$$

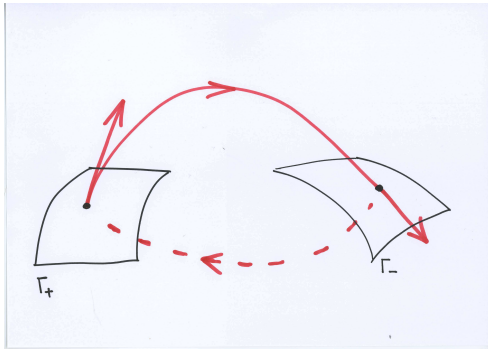
The discrete dynamics change the state instantaneously by the rule

$$F : \Gamma_- \ni [q_-, \dot{q}_-] \mapsto [q_+, \dot{q}_+] \in \Gamma_+$$

where

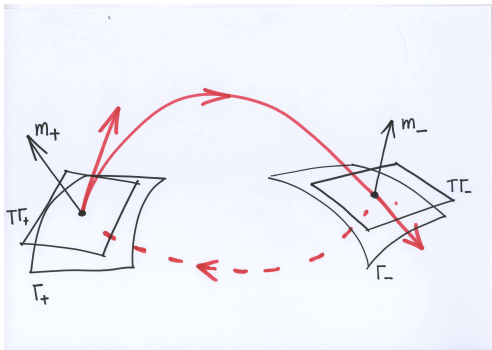
$$q_+ = F_1 q_-, \quad \dot{q}_+ = F_2(q_-) \dot{q}_- \quad \Gamma_+ = \Gamma_-$$

Symmetric gait of a compass biped



- A cycle consists of sub-arc of continuous-in-time motion and the update map of the end point to the beginning.
- One of hypersurfaces Γ_- , Γ_+ is commonly used as Poincare section for analyzing the orbital stability of the gait.

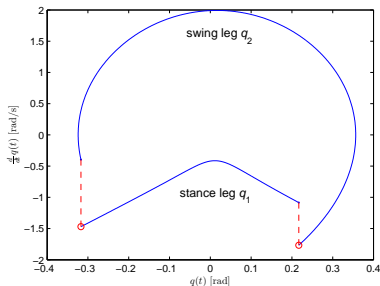
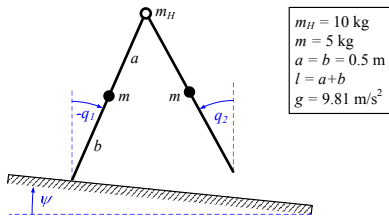
Symmetric gait of a compass biped



Tangent planes $T\Gamma_-$ and $T\Gamma_+$ of Γ_- and Γ_+ at two points of the cycle are used for defining the (numerical) linearization of Poincare map:

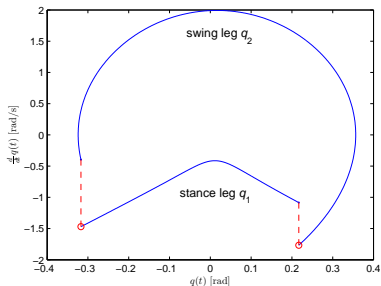
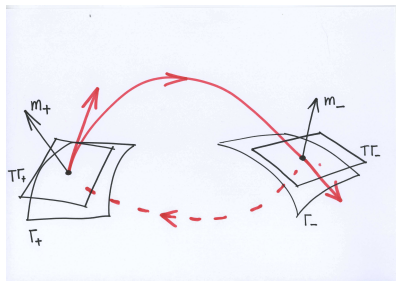
$$z_{k+1} = Az_k, \quad z_{k+1}, z_k \in T\Gamma_+, \quad \dim T\Gamma_+ = (2n - 1) = 3$$

A stable gait for $\psi = 2.87^\circ$



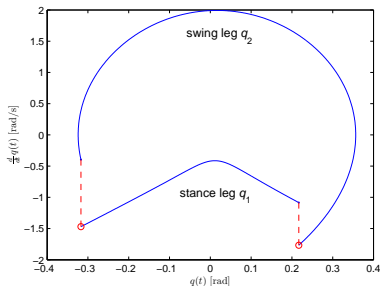
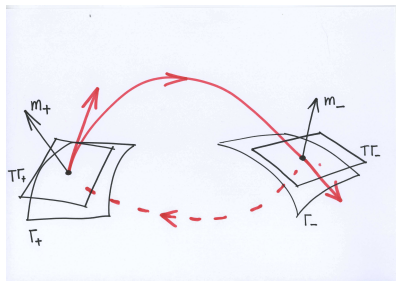
One of stable cycles of the walker shown
on the combined phase portrait

A stable gait for $\psi = 2.87^\circ$



$$\vec{m}_+ = \left[-\sin(q_{1*}(0) + \psi), 0, \sin(q_{2*}(0) + \psi), 0 \right]^T \approx \left[-0.26, 0, 0.26, 0 \right]^T$$

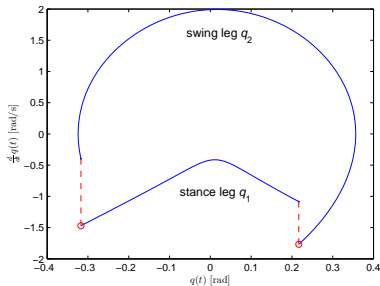
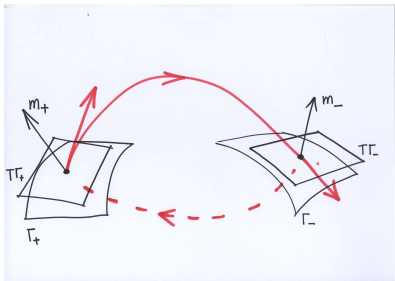
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$$\vec{f}_*(0) = \left[\dot{q}_{1*}(0), \ddot{q}_{1*}(0), \dot{q}_{2*}(0), \ddot{q}_{2*}(0) \right] \approx \left[-1.08, 3.8, -0.39, 11.47 \right]$$

A stable gait for $\psi = 2.87^\circ$

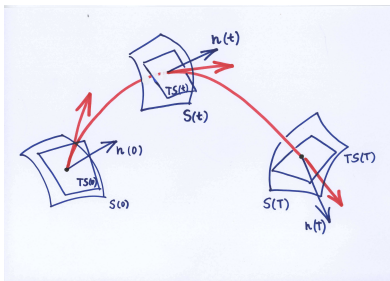
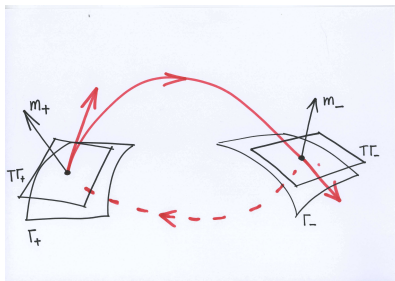


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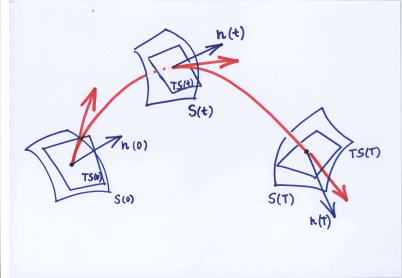
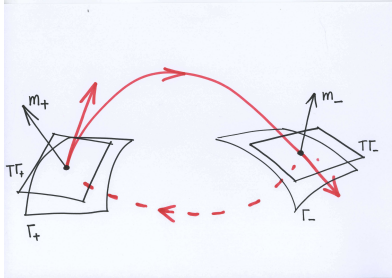
$$\cos \left(\widehat{\vec{m}_+, \vec{f}_*(0)} \right) = \frac{\vec{m}_+ \cdot \vec{f}_*(0)}{|\vec{m}_+| |\vec{f}_*(0)|} \approx 0.18$$

Searching for Alternative Poincare Sections



There are plenty of other choices of transverse sections $S(t)$, $t \in [0, T_p]$, for continuous-in-time arc of the cycle.

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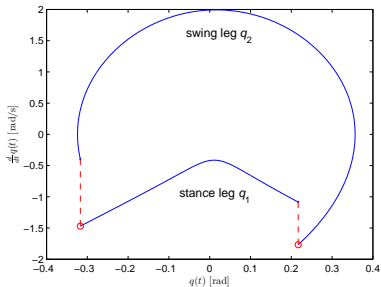


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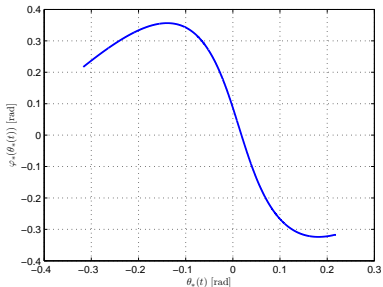
There are special Poincare surfaces $S(t)$: **orthogonal** to the vector field on the cycle at any time

$$\left\{ \vec{f}_*(t) \right\} \perp \left\{ TS(t) \right\}, \quad \forall t \in [0, T_p]$$

Symmetric gait and its nested representation



(a) Phase portrait



(b) The recomputed kinematic relation $q_2 = \phi(q_1)$ between the coordinates on the gait

Transverse coordinates for the gait

Given a motion

$$q_*(t) = [q_{1*}(t), q_{2*}(t)], \quad t \in [0, T_e]$$

and associated virtual holonomic constraint

$$q_{2*}(t) = \phi(q_{1*}(t))$$

New generalized coordinates are θ and y with

$$\theta = q_1, \quad y = q_2 - \phi(\theta)$$

For the system with $x = (q_1, q_2, \dot{q}_1, \dot{q}_2)^T$, $\dim x = 4$, three transverse coordinates x_{\perp} can be defined by

$$x_{\perp} = [I(\theta, \dot{\theta}, \theta_*(0), \dot{\theta}_*(0)), y, \dot{y}]^T$$

where $I(\cdot)$ is the integral of the (α, β, γ) -equation.

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Hybrid transverse linearization for the compass biped gait

Given a cycle $q_*(t) = q_*(t + T_e)$ of the compass-gait walker,

- There is a linear hybrid system

$$\frac{d}{d\tau}z = A(\tau)z, \quad z \in \mathbb{R}^3, \quad A(\tau) = A(\tau + T_e), \quad z_+ = Lz_-$$

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- If it is integrated over period $T_e \approx 0.7324$ s, then we obtain

$$z_{k+1} = A_0 z_k, \quad k = 0, 1, 2, \dots$$

where

$$A_0 = \begin{bmatrix} 0.106396835734031 & 3.885253845507885 & 0.395255324457403 \\ -0.114586152743185 & 0.668714216213521 & 0.062728332694704 \\ 0.488888548158536 & -7.764952916374034 & -0.673609328082623 \end{bmatrix}$$

It gives the linearization of the first-return Poincare map.