

Lecture 3

Constrained Mechanical Systems

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Learning outcomes: Nonlinear mechanical systems with constraints. Classification of constraints. Examples.

1. Systems subject to holonomic constraints

- Example: constrained point-mass dynamics

2. Systems with non-holonomic constraints

- Example: constrained two point-masses dynamics

3. Rigid bodies in contact

- Example: disc rolling and sliding on inclined line

Systems subject to holonomic constraints

Example: point-mass dynamics in excessive coordinates

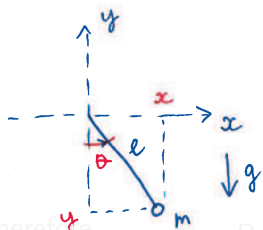
Let a point mass with coordinates (x, y) move in the vertical plane affected by the gravity. Assume that the distance of the point to the origin is kept constant

$$f(x(t), y(t)) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall t.$$

The dynamics of the point are

$$m \cdot \ddot{x} = R_x, \quad m \cdot \ddot{y} = R_y - m \cdot g,$$

where $R = [R_x; R_y]$ is the reaction force due to the constraint.



Example: point-mass dynamics in excessive coordinates

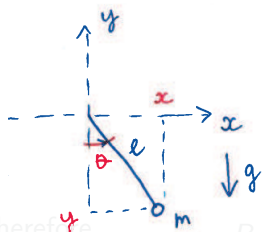
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How to find the reaction force

$$R = R(x, y, \dot{x}, \dot{y})?$$

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where $R = [R_x; R_y]$ is the reaction force due to the constraint.

The constraint $f(\cdot) \equiv 0$ implies that

$$\frac{d}{dt}f = 2x(t) \cdot \dot{x}(t) + 2y(t) \cdot \dot{y}(t) \equiv 0, \quad \forall t.$$

Therefore

$$R_x = \lambda \cdot x, \quad R_y = \lambda \cdot y.$$

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For defining λ consider the 2nd derivative of the constraint $f(\cdot) \equiv 0$

$$\begin{aligned} 0 \equiv \frac{d^2}{dt^2} f &= 2\dot{x}^2 + 2x \cdot \ddot{x} + 2\dot{y}^2 + 2y \cdot \ddot{y} \\ &= 2\dot{x}^2 + 2x \cdot \left(\frac{1}{m} \lambda \cdot x\right) + 2\dot{y}^2 + 2y \cdot \left(\frac{1}{m} \lambda \cdot y - g\right) \\ &= 2\dot{x}^2 + 2\dot{y}^2 + 2\frac{1}{m} \lambda \cdot (x^2 + y^2) - 2y \cdot g \end{aligned}$$

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The Lagrangian multiplier is then equal to

$$\lambda = \frac{m}{l^2} (y \cdot g - \dot{x}^2 - \dot{y}^2)$$

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The point mass dynamics in excessive coordinates (x, y) are

$$m \cdot \ddot{x} = \frac{m}{l^2} (y \cdot g - \dot{x}^2 - \dot{y}^2) \cdot x$$

$$m \cdot \ddot{y} = \frac{m}{l^2} (y \cdot g - \dot{x}^2 - \dot{y}^2) \cdot y - m \cdot g$$

Example: point-mass dynamics in generalized coordinates

To derive the point-mass dynamics with the constraint

$$f(x(t), y(t)) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall t$$

observe that the point's position is determined by the angle θ as

$$x(t) = l \cdot \sin \theta(t), \quad y(t) = -l \cdot \cos \theta(t).$$

The Lagrangian of the system is then

$$\begin{aligned} \mathcal{L} = K - \Pi &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - m \cdot g \cdot y \\ &= \frac{1}{2} m \cdot l^2 \cdot \dot{\theta}^2 + m \cdot g \cdot l \cdot \cos \theta \end{aligned}$$

The dynamics are

$$\begin{aligned} 0 &= \frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right] - \frac{\partial \mathcal{L}}{\partial \theta} = m \cdot l^2 \cdot \ddot{\theta} + m \cdot g \cdot l \cdot \sin \theta \\ &= m \cdot l^2 \cdot \left(\ddot{\theta} + \frac{g}{l} \cdot \sin \theta \right), \end{aligned}$$

which is the equation of the mathematical pendulum.

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Example: point-mass dynamics

The equations written in excessive coordinates (x, y)

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and the equation written in generalized coordinate θ

$$\ddot{\theta} + \frac{g}{l} \cdot \sin \theta = 0\tag{2}$$

represent the dynamics of the same system provided that the initial conditions of both differential equations are appropriately chosen.

However, for mechanical systems with constraints

- the equations of the form (1) can be always derived,
- while the equations of the form (2) might not.

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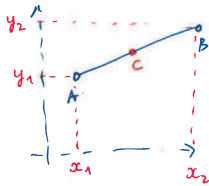
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Systems with non-holonomic constraints

Example: constrained two point-masses dynamics

Consider two point masses of $m = 1$ [kg] each connected by massless rod of length l and moving in the vertical plane.



Constraint No. 1: $(x_1(t) - x_2(t))^2 + (y_1(t) - y_2(t))^2 = l^2, \forall t$

Constraint No. 2: Assume that the velocity of the center of the rod – point C on the plot

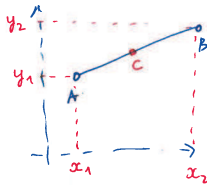
$$\vec{v}_C = \left[\frac{1}{2}(\dot{x}_1 + \dot{x}_2); \frac{1}{2}(\dot{y}_1 + \dot{y}_2) \right]$$

always aligned with the rod.

$$(x_2(t) - x_1(t)) (\dot{y}_1(t) + \dot{y}_2(t)) - (y_2(t) - y_1(t)) (\dot{x}_1(t) + \dot{x}_2(t)) \equiv 0$$

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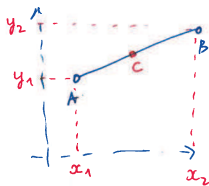
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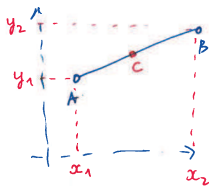
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Example: constrained two point-masses dynamics

The dynamics of the system with coordinates (x_1, y_1, x_2, y_2) are

$$\begin{aligned}\ddot{x}_1 &= R_{x_1}^{(1)} + R_{x_1}^{(2)} & \ddot{x}_2 &= R_{x_2}^{(1)} + R_{x_2}^{(2)} \\ \ddot{y}_1 &= R_{y_1}^{(1)} + R_{y_1}^{(2)} - g & \ddot{y}_2 &= R_{y_2}^{(1)} + R_{y_2}^{(2)} - g\end{aligned}$$

Here $R^{(1)}$ and $R^{(2)}$ are the reaction forces due to constraints

$$R^{(1)} = [R_{x_1}^{(1)}; R_{y_1}^{(1)}; R_{x_2}^{(1)}; R_{y_2}^{(1)}], \quad R^{(2)} = [R_{x_1}^{(2)}; R_{y_1}^{(2)}; R_{x_2}^{(2)}; R_{y_2}^{(2)}]$$

Components of the reaction forces are determined from the assumption that such forces do not dissipate or increase the energy of the system along its motions

$$R_{x_1}^{(i)} \cdot \dot{x}_1 + R_{y_1}^{(i)} \cdot \dot{y}_1 + R_{x_2}^{(i)} \cdot \dot{x}_2 + R_{y_2}^{(i)} \cdot \dot{y}_2 \equiv 0, \quad i = 1, 2.$$

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Observation: The system dynamics are written in 4 excessive coordinates and have 2 constraints. But one cannot reduce a number of coordinates to 2 and derive the dynamics! ☹️☹️☹️

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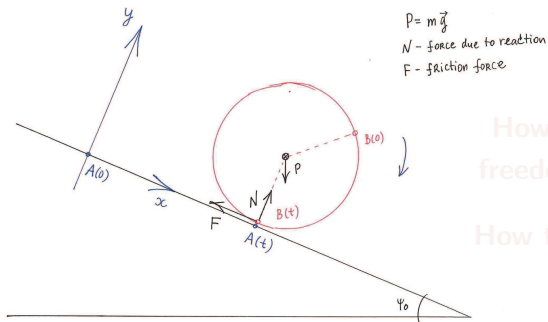
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Observation: The system dynamics are integrable in the sense that all the solutions can be found explicitly! ☺☺☺ see the homework!

Rigid bodies in contact

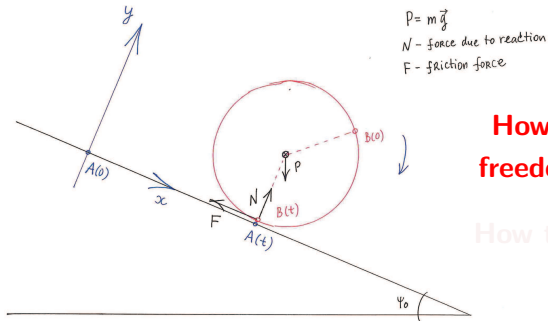
Example: disc rolling on inclined line



How many degrees of freedom of the system?

How to derive equations of motion?

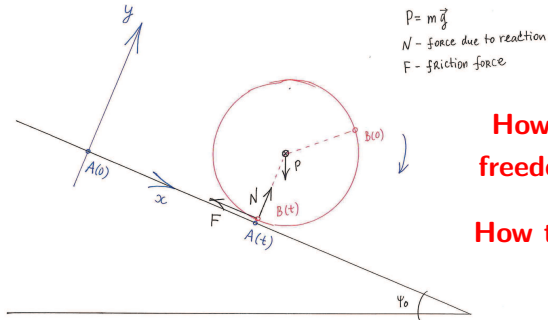
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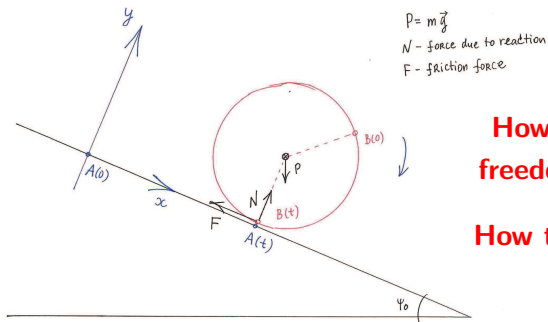
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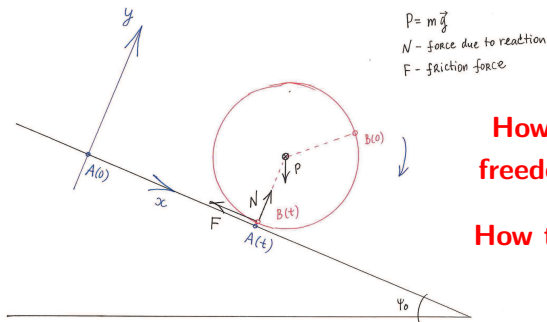
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How to derive equations of motion?

Equations of motion of rigid body are

- Dynamics of centre of mass:
$$\frac{d}{dt} (m\vec{v}_C) = \sum_i \vec{F}_{\text{external}}^{(i)}$$

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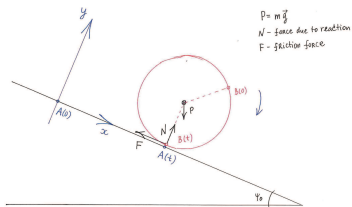
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Equations of motion of rigid body are

- Dynamics of centre of mass:
$$\frac{d}{dt} (m\vec{v}_C) = \sum_i \vec{F}_{\text{external}}^{(i)}$$

- Angular momentum's rate of change:
$$\frac{d}{dt} (\vec{L}) = \sum_i \vec{M}_{\text{external}}^{(i)}$$

Example: disc rolling on inclined line



The dynamics of center of mass \vec{r}_C are

$$M \begin{bmatrix} \ddot{x}_C \\ \ddot{y}_C \\ \ddot{z}_C \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} + \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix} + \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

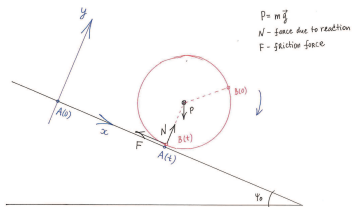
Relative to the origin of the inertia frame, angular momentum is

$$\vec{L}^O := \sum_i \vec{R}_i \times m_i \frac{d}{dt} \vec{R}_i$$

The rate of change of \vec{L}^O is then

$$\frac{d}{dt} \vec{L}^O := \sum_i \frac{d}{dt} \vec{R}_i \times m_i \frac{d}{dt} \vec{R}_i + \sum_i \vec{R}_i \times m_i \frac{d^2}{dt^2} \vec{R}_i$$

Example: disc rolling on inclined line



The dynamics of center of mass \vec{r}_C are

$$M \begin{bmatrix} \ddot{x}_C \\ \ddot{y}_C \\ 0 \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \\ 0 \end{bmatrix} + \begin{bmatrix} N_x \\ N_y \\ 0 \end{bmatrix} + \begin{bmatrix} F_x \\ F_y \\ 0 \end{bmatrix}$$

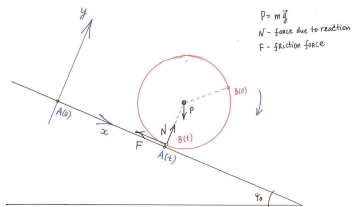
Relative to the origin of the inertia frame, angular momentum is

$$\vec{L}^O := \sum_i \vec{R}_i \times m_i \frac{d}{dt} \vec{R}_i$$

The rate of change of \vec{L}^O is then

$$\frac{d}{dt} \vec{L}^O := \sum_i \frac{d}{dt} \vec{R}_i \times m_i \frac{d}{dt} \vec{R}_i + \sum_i \vec{R}_i \times m_i \frac{d^2}{dt^2} \vec{R}_i$$

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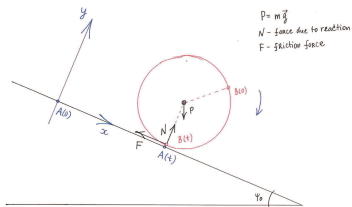
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Example: disc rolling on inclined line



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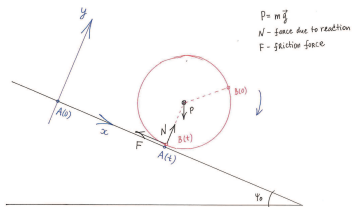
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Example: disc rolling on inclined line



The dynamics of center of mass \vec{r}_C are

$$M\ddot{x}_C = m \cdot g \cdot \sin \psi_0 - F$$

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$$0 = N - m \cdot g \cdot \cos \psi_0$$

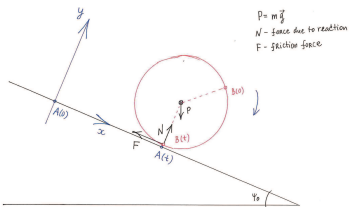
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Example: disc rolling on inclined line



The dynamics of angular momentum $\vec{L}(\cdot)$ depends on a choice of the origin it is computed about!

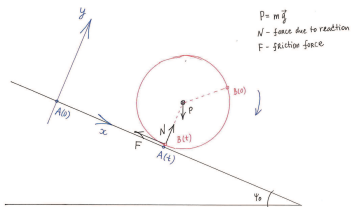
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Example: disc rolling on inclined line



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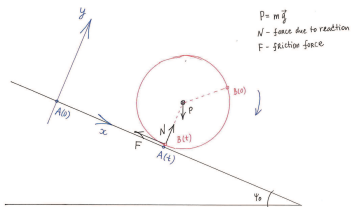
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The rate of change of \vec{L}^O is then

$$\frac{d}{dt} \vec{L}^O := \sum_i \frac{d}{dt} \vec{R}_i \times m_i \frac{d}{dt} \vec{R}_i + \sum_i \vec{R}_i \times m_i \frac{d^2}{dt^2} \vec{R}_i$$

Example: disc rolling on inclined line



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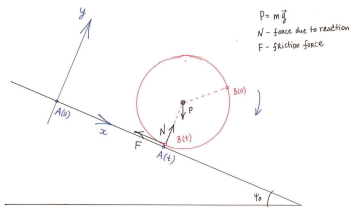
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Example: disc rolling on inclined line



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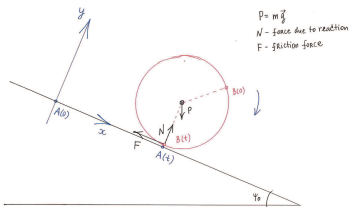
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The rate of change of \vec{L}^O is then

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Example: disc rolling on inclined line



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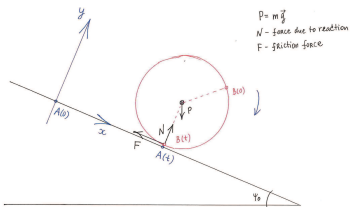
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Relative to **the centre of mass**, angular momentum is

$$\vec{L}^C := \sum_i \vec{r}_i \times m_i \frac{d}{dt} \vec{R}_i = \sum_i \vec{r}_i \times m_i \frac{d}{dt} (\vec{R}_C + \vec{r}_i)$$

Example: disc rolling on inclined line



The dynamics of angular momentum $\vec{L}(\cdot)$ depends on a choice of the origin it is computed about!

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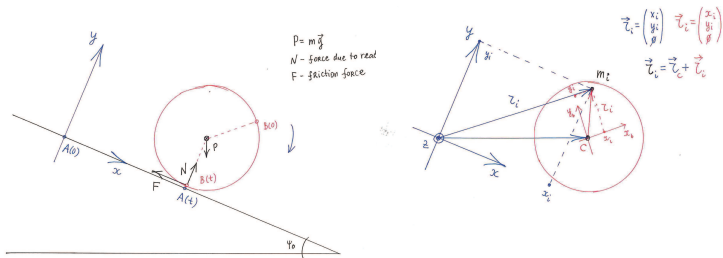
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$$\frac{d}{dt} \vec{L}^C = \sum_i \frac{d}{dt} \vec{r}_i \times m_i \frac{d}{dt} \vec{R}_i + \sum_i \vec{r}_i \times m_i \frac{d^2}{dt^2} \vec{R}_i = \sum_k \vec{r}_k \times \vec{F}_{\text{external}}^{(k)}$$

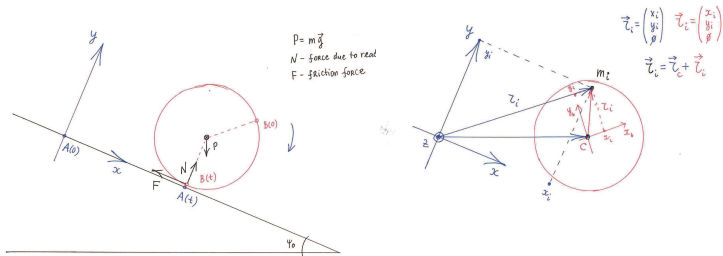
Example: disc rolling on inclined line



$$\vec{r}_i^O = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta} \vec{r}_i^D, \quad \vec{v}_i = \frac{d}{dt} \vec{R}_C + \frac{d}{dt} \vec{r}_i^O$$

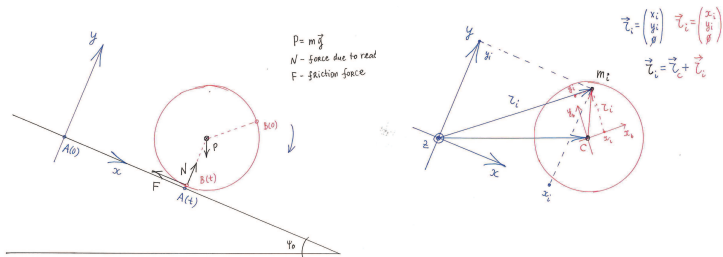
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Example: disc rolling on inclined line



$$\vec{r}_i^O = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{= R_{z,\theta}} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix}, \quad \vec{v}_i = \frac{d}{dt} \vec{R}_C + \frac{d}{dt} \vec{r}_i^O$$

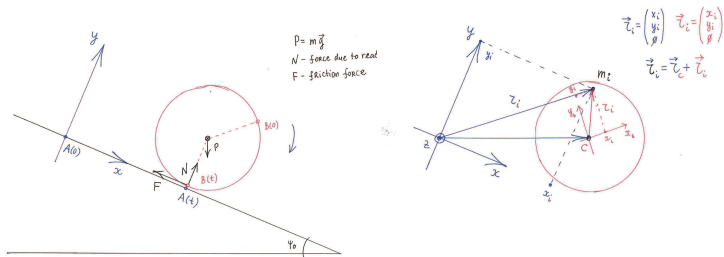
Example: disc rolling on inclined line



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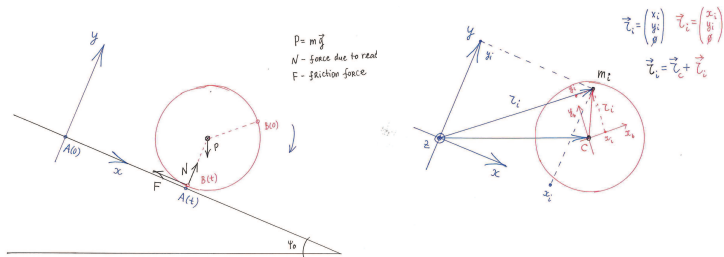
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$$\vec{L}^C = \sum_i \vec{r}_i^O \times m_i \vec{v}_i = \sum_i R_{z,\theta} \vec{r}_i^D \times m_i \left(\frac{d}{dt} \vec{R}_C + \frac{d}{dt} [R_{z,\theta} \vec{r}_i^D] \right)$$

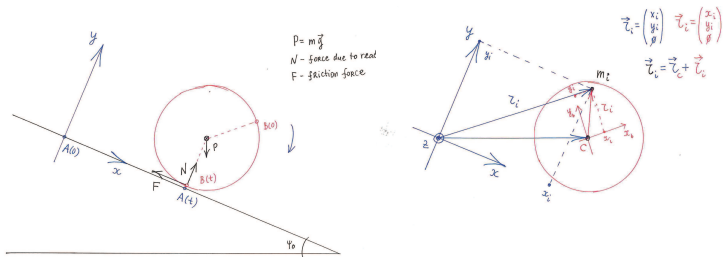
Example: disc rolling on inclined line



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$$\begin{aligned} \vec{L}^C &= \sum_i \vec{r}_i^O \times m_i \vec{v}_i = \sum_i R_{z,\theta} \vec{r}_i^D \times m_i \left(\frac{d}{dt} \vec{R}_C + \frac{d}{dt} [R_{z,\theta} \vec{r}_i^D] \right) \\ &= \sum_i \vec{r}_i^O \times m_i \frac{d}{dt} \vec{R}_C + \sum_i R_{z,\theta} \vec{r}_i^D \times m_i \frac{d}{dt} [R_{z,\theta} \vec{r}_i^D] \end{aligned}$$

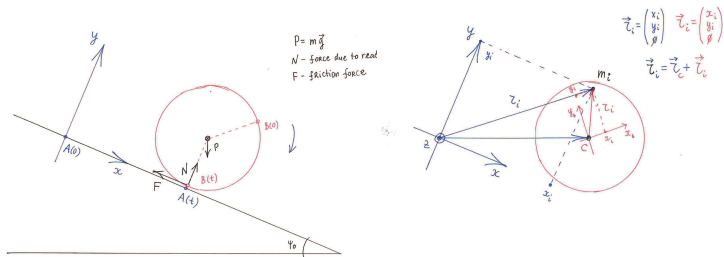
Example: disc rolling on inclined line



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$$\begin{aligned} \vec{L}^C &= \sum_i \vec{r}_i^O \times m_i \vec{v}_i = \sum_i R_{z,\theta} \vec{r}_i^D \times m_i \left(\frac{d}{dt} \vec{R}_C + \frac{d}{dt} [R_{z,\theta} \vec{r}_i^D] \right) \\ &= \left(\sum_i m_i \vec{r}_i^O \right) \times \frac{d}{dt} \vec{R}_C + \sum_i R_{z,\theta} \vec{r}_i^D \times m_i \frac{d}{dt} [R_{z,\theta} \vec{r}_i^D] \end{aligned}$$

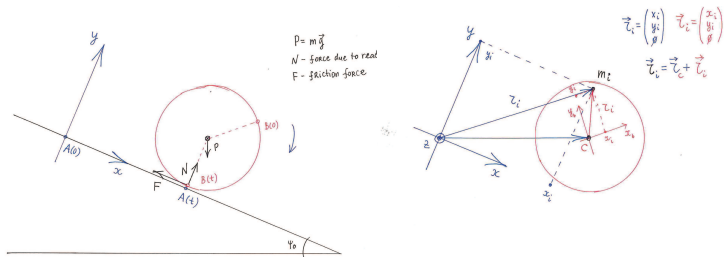
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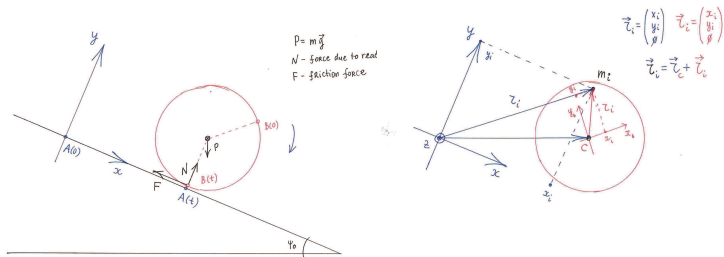
Example: disc rolling on inclined line



$$\vec{r}_i^O = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z, \theta} \vec{r}_i^D, \quad \vec{v}_i = \frac{d}{dt} \vec{R}_C + \frac{d}{dt} \vec{r}_i^O$$

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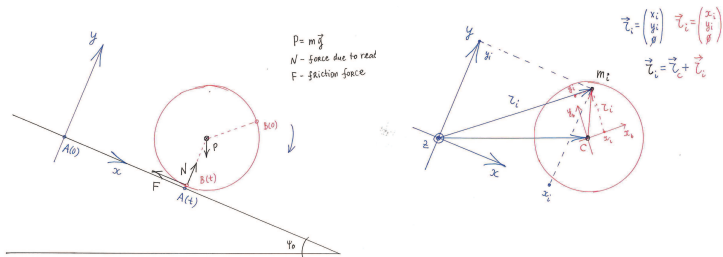
Example: disc rolling on inclined line



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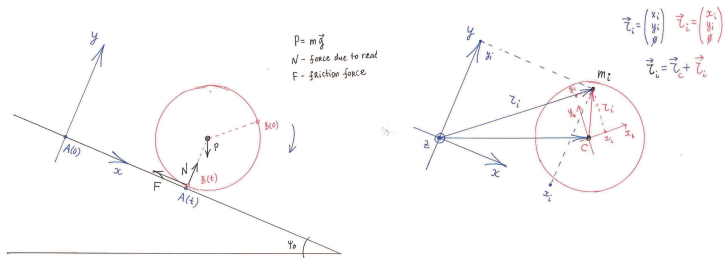
Example: disc rolling on inclined line



$$\vec{r}_i^O = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z, \theta} \vec{r}_i^D, \quad \vec{v}_i = \frac{d}{dt} \vec{R}_C + \frac{d}{dt} \vec{r}_i^O$$

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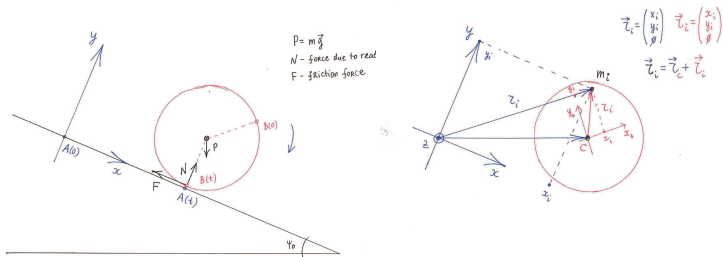
Example: disc rolling on inclined line



The rate of change of angular momentum \vec{L}^C is then

$$\frac{d}{dt} \begin{bmatrix} 0 \\ 0 \\ J_z \dot{\theta} \end{bmatrix} = \vec{r}_P^O \times \vec{P} + \vec{r}_N^O \times \vec{N} + \vec{r}_F^O \times \vec{F}$$

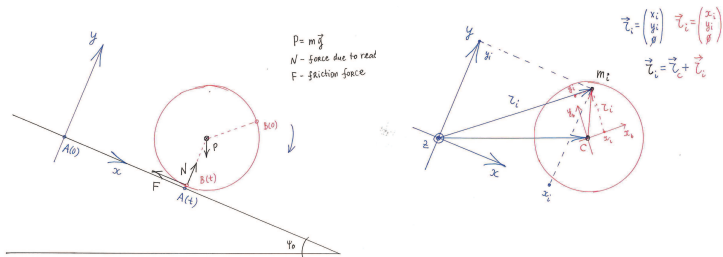
Example: disc rolling on inclined line



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$$\begin{aligned}
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 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \vec{P} + \begin{bmatrix} 0 \\ -R_d \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ N_y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -R_d \\ 0 \end{bmatrix} \times \begin{bmatrix} -F_x \\ 0 \\ 0 \end{bmatrix}
 \end{aligned}$$

Example: disc rolling on inclined line

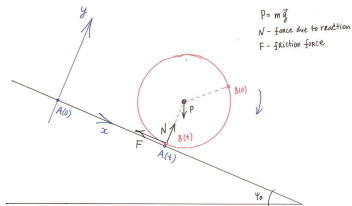


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$$\frac{d}{dt} \begin{bmatrix} 0 \\ 0 \\ J_z \dot{\theta} \end{bmatrix} = \vec{r}_P^O \times \vec{P} + \vec{r}_N^O \times \vec{N} + \vec{r}_F^O \times \vec{F} = \begin{bmatrix} 0 \\ 0 \\ -R_d \cdot F_x \end{bmatrix}$$

Example: disc rolling on inclined line

The dynamics of the system are



$$M \cdot \ddot{x}_C = m \cdot g \cdot \sin \psi_0 - F$$

$$\ddot{y}_C = 0$$

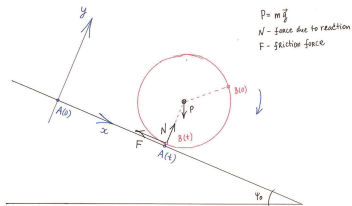
$$0 = N - m \cdot g \cdot \cos \psi_0$$

$$J_z \cdot \ddot{\theta} = -R_d \cdot F$$

How many unknown variables?

Example: disc rolling on inclined line

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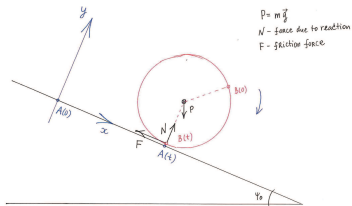
$$0 = N - m \cdot g \cdot \cos \psi_0$$

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Example: disc rolling on inclined line

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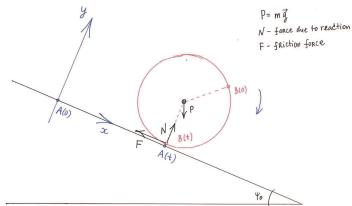
$$0 = N - m \cdot g \cdot \cos \psi_0$$

$$J_z \cdot \ddot{\theta} = -R_d \cdot F$$

How many unknown variables? **Three**, they are: $x_C(\cdot)$, $\theta(\cdot)$, $F(\cdot)$

Example: disc rolling on inclined line

The dynamics of the system are



$$M \cdot \ddot{x}_C = m \cdot g \cdot \sin \psi_0 - F$$

$$\ddot{y}_C = 0$$

$$0 = N - m \cdot g \cdot \cos \psi_0$$

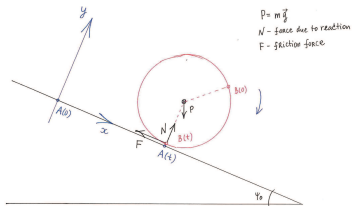
$$J_z \cdot \ddot{\theta} = -R_d \cdot F$$

How many unknown variables? **Three**, they are: $x_C(\cdot)$, $\theta(\cdot)$, $F(\cdot)$

How many nontrivial equations?

Example: disc rolling on inclined line

The dynamics of the system are



$$M \cdot \ddot{x}_C = m \cdot g \cdot \sin \psi_0 - F$$

$$\ddot{y}_C = 0$$

$$0 = N - m \cdot g \cdot \cos \psi_0$$

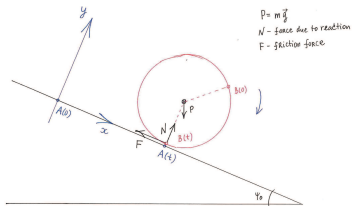
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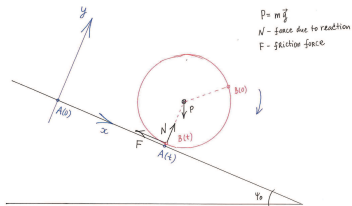
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How many nontrivial equations? **Two!**

We need one more equation for the system!

Example: disc rolling on inclined line

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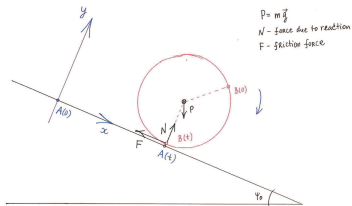
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If the disc slides for some time interval $\Rightarrow F(t) = \mu \cdot N$

Example: disc rolling on inclined line

The dynamics of the system are



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$$J_z \cdot \ddot{\theta} = -R_d \cdot F$$

How many unknown variables? **Three**, they are: $x_C(\cdot)$, $\theta(\cdot)$, $F(\cdot)$

How many nontrivial equations? **Two!**

If the disc slides for some time interval $\Rightarrow F(t) = \mu \cdot N$

If the disc rolls without sliding for some interval

$$\Rightarrow x_C(t) - x_C(0) = -R_d \cdot (\theta(t) - \theta(0))$$