Lecture 3

Constrained Mechanical Systems

Anton Shiriaev
LIRMM, Montpellier
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Learning outcomes: Nonlinear mechanical systems with constraints. Classification of constraints. Examples.

Outline

1. Systems subject to holonomic constraints
   - Example: constrained point-mass dynamics
2. Systems with non-holonomic constraints
   - Example: constrained two point-masses dynamics
3. Rigid bodies in contact
   - Example: disc rolling and sliding on inclined line

Systems subject to holonomic constraints

Example: point-mass dynamics in excessive coordinates

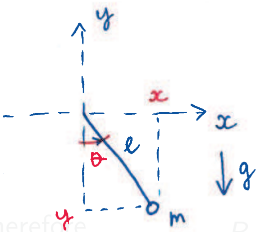
Let a point mass with coordinates (x, y) move in the vertical plane affected by the gravity. Assume that the distance of the point to the origin is kept constant

$$f\,(x(t), y(t)) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall\, t.$$

The dynamics of the point are

$$m \cdot \ddot{x} = R_x, \quad m \cdot \ddot{y} = R_y - m \cdot g,$$

where $R = [R_x;\ R_y]$ is the reaction force due to the constraint.



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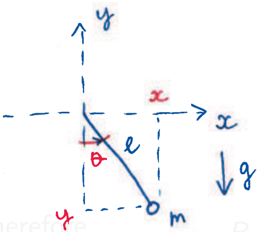

How to find the reaction force
$R = R(x, y, \dot{x}, \dot{y})$**?**

Therefore $R_x = \lambda \cdot x, \quad R_y = \lambda \cdot y.$

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The constraint $f(\cdot) \equiv 0$ implies that

$$\tfrac{d}{dt} f = 2x(t) \cdot \dot{x}(t) + 2y(t) \cdot \dot{y}(t) \equiv 0, \quad \forall\, t.$$

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For defining λ consider the 2^{nd} derivative of the constraint $f(\cdot) \equiv 0$

$$\begin{aligned} 0 \equiv \tfrac{d^2}{dt^2} f &= 2\dot{x}^2 + 2x \cdot \ddot{x} + 2\dot{y}^2 + 2y \cdot \ddot{y} \\ &= 2\dot{x}^2 + 2x \cdot (\tfrac{1}{m}\lambda \cdot x) + 2\dot{y}^2 + 2y \cdot (\tfrac{1}{m}\lambda \cdot y - g) \\ &= 2\dot{x}^2 + 2\dot{y}^2 + 2\tfrac{1}{m}\lambda \cdot (x^2 + y^2) - 2y \cdot g \end{aligned}$$

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Therefore

$$R_x = \lambda \cdot x, \quad R_y = \lambda \cdot y.$$

The Lagrangian multiplier is then equal to

$$\lambda = \frac{m}{l^2}\left(y \cdot g - \dot{x}^2 - \dot{y}^2\right)$$

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The point mass dynamics in excessive coordinates (x, y) are

$$\begin{aligned} m \cdot \ddot{x} &= \frac{m}{l^2}\left(y \cdot g - \dot{x}^2 - \dot{y}^2\right) \cdot x \\ m \cdot \ddot{y} &= \frac{m}{l^2}\left(y \cdot g - \dot{x}^2 - \dot{y}^2\right) \cdot y - m \cdot g \end{aligned}$$

Example: point-mass dynamics in generalized coordinates

To derive the point-mass dynamics with the constraint

$$f\left(x(t), y(t)\right) := x(t)^2 + y(t)^2 - l^2 \equiv 0, \quad \forall\, t$$

observe that the point's position is determined by the angle θ as

$$x(t) = l \cdot \sin\theta(t), \quad y(t) = -l \cdot \cos\theta(t).$$

The Lagrangian of the system is then

$$\begin{aligned} \mathcal{L} = K - \Pi &= \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - m \cdot g \cdot y \\ &= \tfrac{1}{2} m \cdot l^2 \cdot \dot{\theta}^2 + m \cdot g \cdot l \cdot \cos\theta \end{aligned}$$

The dynamics are

$$\begin{aligned} 0 = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] - \frac{\partial \mathcal{L}}{\partial \theta} &= m \cdot l^2 \cdot \ddot{\theta} + m \cdot g \cdot l \cdot \sin\theta \\ &= m \cdot l^2 \cdot \left(\ddot{\theta} + \frac{g}{l} \cdot \sin\theta\right), \end{aligned}$$

which is the equation of the mathematical pendulum.

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Example: point-mass dynamics

The equations written in excessive coordinates (x, y)

$$\begin{aligned} m \cdot \ddot{x} &= \tfrac{m}{l^2}\left(y \cdot g - \dot{x}^2 - \dot{y}^2\right) \cdot x \\ m \cdot \ddot{y} &= \tfrac{m}{l^2}\left(y \cdot g - \dot{x}^2 - \dot{y}^2\right) \cdot y - m \cdot g \end{aligned} \qquad (1)$$

and the equation written in generalized coordinate θ

$$\ddot{\theta} + \frac{g}{l} \cdot \sin\theta = 0 \qquad (2)$$

represent the dynamics of the same system provided that the initial conditions of both differential equations are appropriately chosen.

However, for mechanical systems with constraints

- the equations of the form (1) can be always derived,
- while the equations of the form (2) might not.

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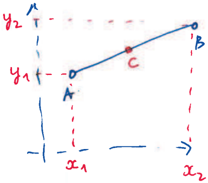
Systems with non-holonomic constraints

Example: constrained two point-masses dynamics

Consider two point masses of $m = 1$ [kg] each connected by massless rod of length l and moving in the vertical plane.



Constraint No. 1: $\Big(x_1(t) - x_2(t)\Big)^2 + \Big(y_1(t) - y_2(t)\Big)^2 = l^2, \ \forall\, t$

Constraint No. 2: Assume that the velocity of the center of the rod – point C on the plot

$$\vec{v}_C = \left[\tfrac{1}{2}(\dot{x}_1 + \dot{x}_2);\ \tfrac{1}{2}(\dot{y}_1 + \dot{y}_2)\right]$$

always aligned with the rod.

$$\Big(x_2(t) - x_1(t)\Big)\Big(\dot{y}_1(t) + \dot{y}_2(t)\Big) - \Big(y_2(t) - y_1(t)\Big)\Big(\dot{x}_1(t) + \dot{x}_2(t)\Big) = 0$$

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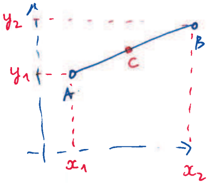

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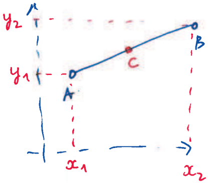

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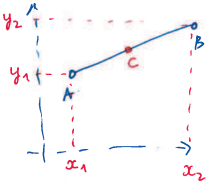

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Example: constrained two point-masses dynamics

The dynamics of the system with coordinates (x_1, y_1, x_2, y_2) are

$$\begin{aligned} \ddot{x}_1 &= R^{(1)}_{x_1} + R^{(2)}_{x_1} & \qquad \ddot{x}_2 &= R^{(1)}_{x_2} + R^{(2)}_{x_2} \\ \ddot{y}_1 &= R^{(1)}_{y_1} + R^{(2)}_{y_1} - g & \qquad \ddot{y}_2 &= R^{(1)}_{y_2} + R^{(2)}_{y_2} - g \end{aligned}$$

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Components of of the reaction forces are determined from the assumption that such forces do not dissipate or increase the energy of the system along its motions

$$R^{(i)}_{x_1} \cdot \dot{x}_1 + R^{(i)}_{y_1} \cdot \dot{y}_1 + R^{(i)}_{x_2} \cdot \dot{x}_2 + R^{(i)}_{y_2} \cdot \dot{y}_2 \equiv 0, \quad i = 1, 2.$$

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Observation: The system dynamics are written in 4 excessive coordinates and have 2 constraints. But one cannot reduce a number of coordinates to 2 and derive the dynamics! ☹☹☹

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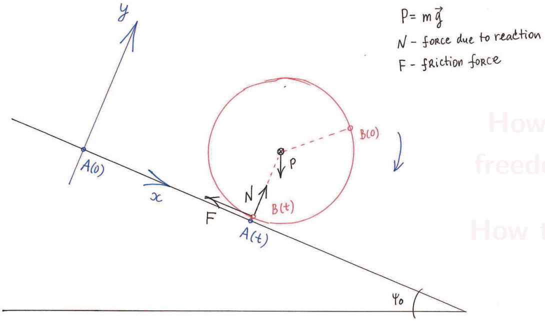
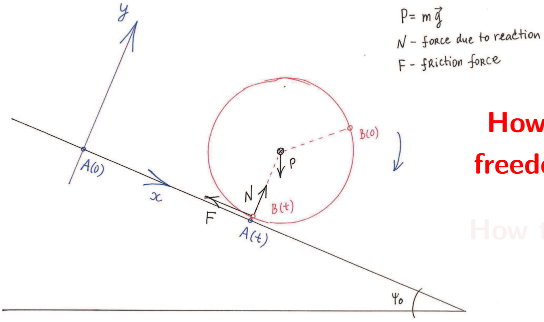
Observation: The system dynamics are integrable in the sense that all the solutions can be found explicitly! ☺☺☺ see the homework!

Rigid bodies in contact



How many degrees of freedom of the system?

How to derive equations of motion?



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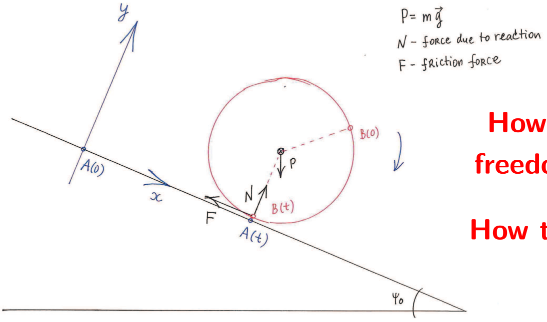
Example: disc rolling on inclined line



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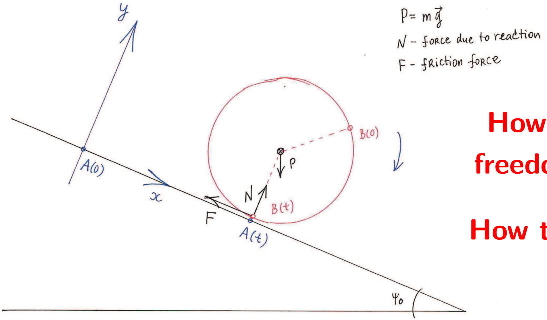

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Equations of motion of rigid body are

- Dynamics of centre of mass: $\frac{d}{dt}\left(m\vec{v}_C\right) = \sum_i \vec{F}^{(i)}_{external}$

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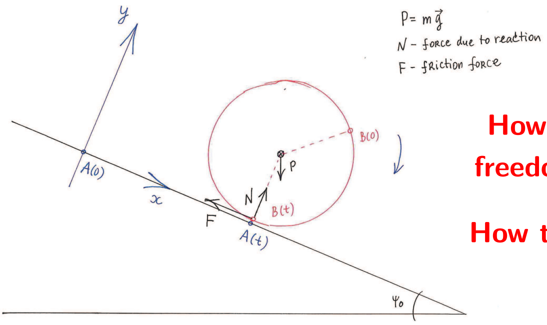

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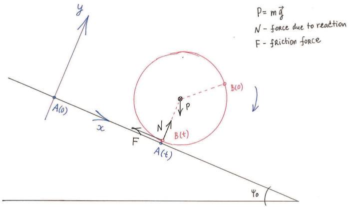
Equations of motion of rigid body are

- Dynamics of centre of mass: $\frac{d}{dt}\left(m\vec{v}_C\right) = \sum_i \vec{F}^{(i)}_{external}$
- Angular momentum's rate of change: $\frac{d}{dt}\left(\vec{L}\right) = \sum_i \vec{M}^{(i)}_{external}$

Example: disc rolling on inclined line



The dynamics of center of mass $\vec{r}_C$ are

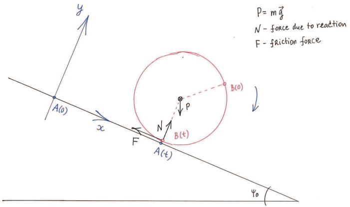
$$M \begin{bmatrix} \ddot{x}_C \\ \ddot{y}_C \\ \ddot{z}_C \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} + \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix} + \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

Relative to the origin of the inertia frame, angular momentum is

$$\vec{L}^O \quad := \quad \sum_i \vec{R}_i \times m_i \frac{d}{dt} \vec{R}_i$$

The rate of change of $\vec{L}^O$ is then

$$\frac{d}{dt}\vec{L}^O \quad := \quad \sum_i \frac{d}{dt}\vec{R}_i \times m_i \frac{d}{dt}\vec{R}_i + \sum_i \vec{R}_i \times m_i \frac{d^2}{dt^2}\vec{R}_i$$



The dynamics of center of mass $\vec{r}_C$ are

$$M \begin{bmatrix} \ddot{x}_C \\ \ddot{y}_C \\ 0 \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \\ 0 \end{bmatrix} + \begin{bmatrix} N_x \\ N_y \\ 0 \end{bmatrix} + \begin{bmatrix} F_x \\ F_y \\ 0 \end{bmatrix}$$

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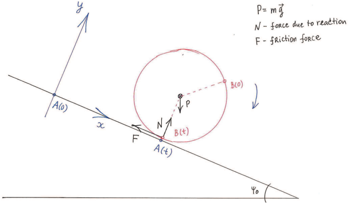

The dynamics of center of mass $\vec{r}_C$ are

$$M \begin{bmatrix} \ddot{x}_C \\ \ddot{y}_C \\ 0 \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ N_y \\ 0 \end{bmatrix} + \begin{bmatrix} F_x \\ 0 \\ 0 \end{bmatrix}$$

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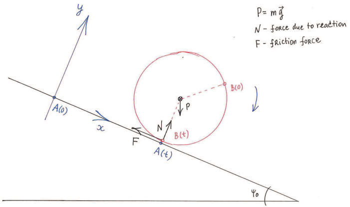

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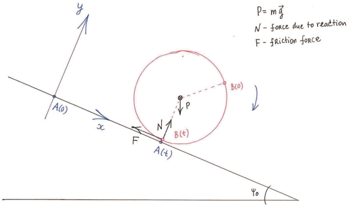
Example: disc rolling on inclined line



The dynamics of center of mass $\vec{r}_C$ are

$$
\begin{aligned}
M\ddot{x}_C &= m \cdot g \cdot \sin\psi_0 - F \\
\ddot{y}_C &= 0 \\
0 &= N - m \cdot g \cdot \cos\psi_0
\end{aligned}
$$

Relative to the origin of the inertia frame, angular momentum is

$$\vec{L}^O := \sum_i \vec{R}_i \times m_i \frac{d}{dt}\vec{R}_i$$

The rate of change of $\vec{L}^O$ is then

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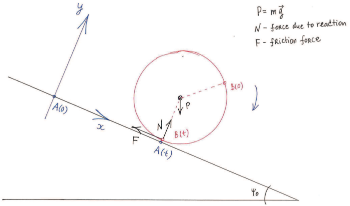

The dynamics of angular momentum $\vec{L}(\cdot)$ depends on a choice of the origin it is computed about!

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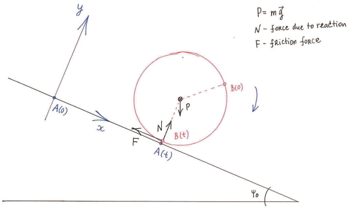

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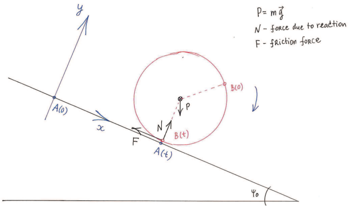

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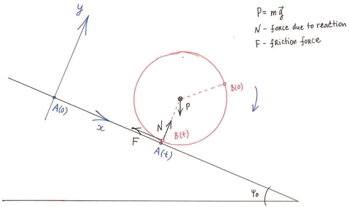
Example: disc rolling on inclined line



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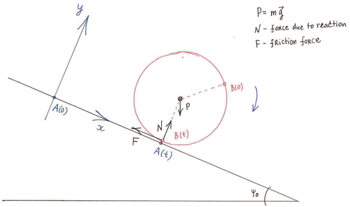
$$= \;\sum_i \vec{R}_i \times m_i \frac{d^2}{dt^2}\vec{R}_i = \sum_i \vec{R}_i \times \vec{F}^{(i)}$$



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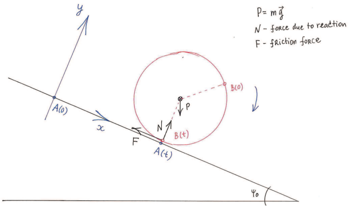
Example: disc rolling on inclined line



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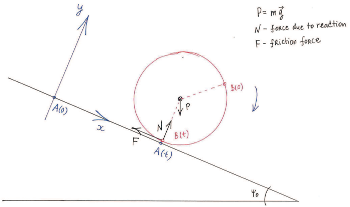
Relative to the centre of mass, angular momentum is

$$\vec{L}^C \quad := \quad \sum_i \vec{r}_i \times m_i \frac{d}{dt} \vec{R}_i = \sum_i \vec{r}_i \times m_i \frac{d}{dt} \left(\vec{R}_C + \vec{r}_i \right)$$

Example: disc rolling on inclined line



The dynamics of angular momentum $\vec{L}(\cdot)$ depends on a choice of the origin it is computed about!

Relative to the origin of the inertia frame, angular momentum is

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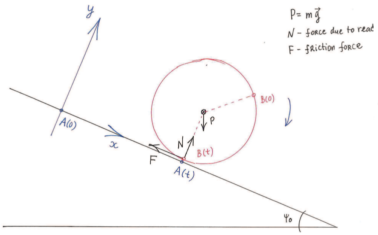
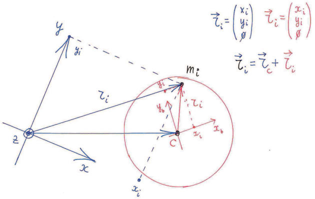
$$\vec{L}^C \; := \; \sum_i \vec{r}_i \times m_i \frac{d}{dt}\vec{R}_i = \sum_i \vec{r}_i \times m_i \frac{d}{dt}\left(\vec{R}_C + \vec{r}_i\right)$$

$$\frac{d}{dt}\vec{L}^C = \sum_i \frac{d}{dt}\vec{r}_i \times m_i \frac{d}{dt}\vec{R}_i + \sum_i \vec{r}_i \times m_i \frac{d^2}{dt^2}\vec{R}_i = \sum_k \vec{r}_k \times \vec{F}^{(k)}_{external}$$

Example: disc rolling on inclined line

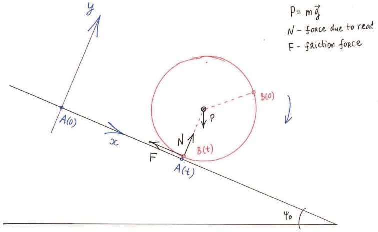


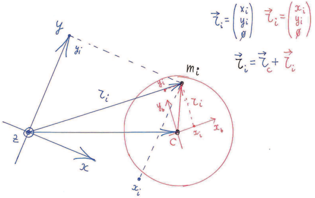


$$\vec{r}_i^{\,O} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta}\vec{r}_i^{\,D}, \quad \vec{v}_i = \frac{d}{dt}\vec{R}_C + \frac{d}{dt}\vec{r}_i^{\,O}$$

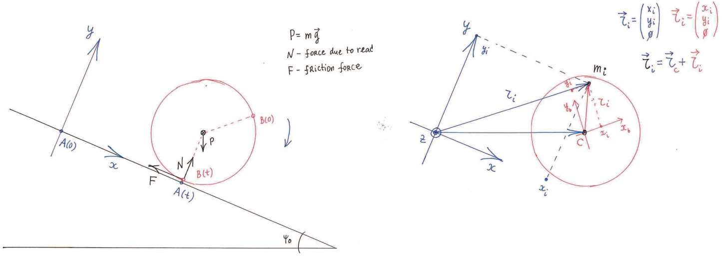
$$\vec{L}^C = \sum_i \vec{r}_i^{\,O} \times m_i \vec{v}_i$$

Example: disc rolling on inclined line





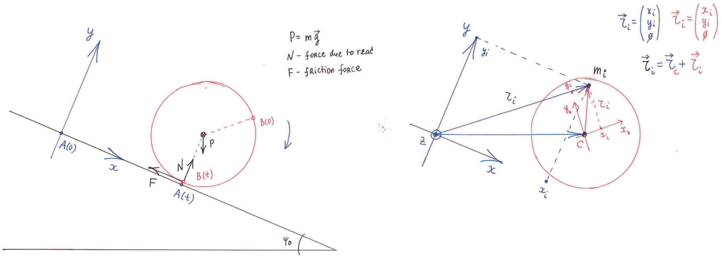
$$\vec{r_i}^{O} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{= R_{z,\theta}} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix}, \qquad \vec{v_i} = \frac{d}{dt}\vec{R}_C + \frac{d}{dt}\vec{r_i}^{O}$$



$$\vec{r_i}^{\,O} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta}\vec{r_i}^{\,D}, \quad \vec{v_i} = \frac{d}{dt}\vec{R}_C + \frac{d}{dt}\vec{r_i}^{\,O}$$

$$\vec{L}^C = \sum_i \vec{r_i}^{\,O} \times m_i \vec{v_i}$$

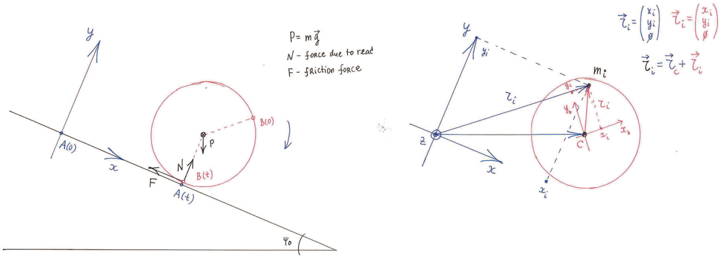
Example: disc rolling on inclined line



$$\vec{r}_i^{\,O} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta}\vec{r}_i^{\,D}, \quad \vec{v}_i = \frac{d}{dt}\vec{R}_C + \frac{d}{dt}\vec{r}_i^{\,O}$$

$$\vec{L}^C \;=\; \sum_i \vec{r}_i^{\,O} \times m_i \vec{v}_i = \sum_i R_{z,\theta}\vec{r}_i^{\,D} \times m_i \left(\frac{d}{dt}\vec{R}_C + \frac{d}{dt}\left[R_{z,\theta}\vec{r}_i^{\,D} \right] \right)$$

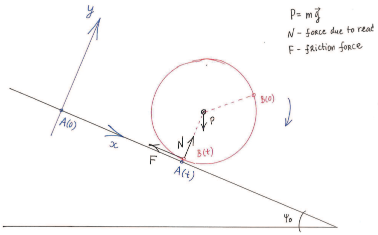
Example: disc rolling on inclined line

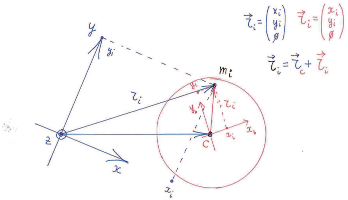


$$\vec{r_i}^O = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta}\vec{r_i}^D, \quad \vec{v_i} = \frac{d}{dt}\vec{R}_C + \frac{d}{dt}\vec{r_i}^O$$

$$\begin{aligned} \vec{L}^C &= \sum_i \vec{r_i}^O \times m_i \vec{v_i} = \sum_i R_{z,\theta}\vec{r_i}^D \times m_i \left(\frac{d}{dt}\vec{R}_C + \frac{d}{dt}\left[R_{z,\theta}\vec{r_i}^D\right]\right) \\ &= \sum_i \vec{r_i}^O \times m_i \frac{d}{dt}\vec{R}_C + \sum_i R_{z,\theta}\vec{r_i}^D \times m_i \frac{d}{dt}\left[R_{z,\theta}\vec{r_i}^D\right] \end{aligned}$$

Example: disc rolling on inclined line





$$\color{red}{\vec{r}_i}^{\,O} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta}\vec{r}_i^{\,D}, \quad \vec{v}_i = \frac{d}{dt}\vec{R}_C + \frac{d}{dt}\vec{r}_i^{\,O}$$

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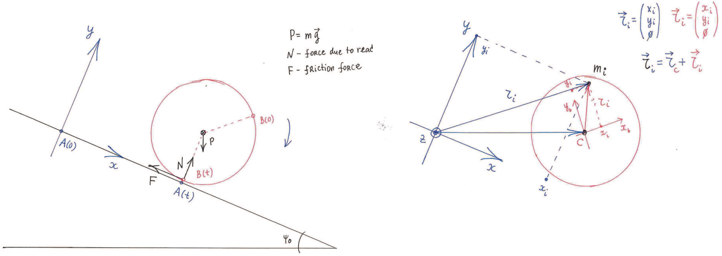

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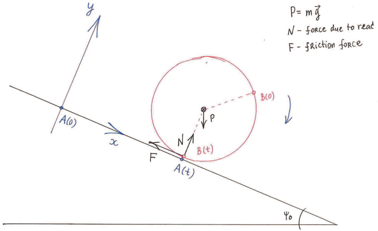
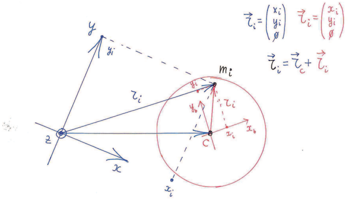
Example: disc rolling on inclined line





$$\vec{r_i}^{O} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta}\vec{r_i}^{D}, \quad \vec{v_i} = \frac{d}{dt}\vec{R}_C + \frac{d}{dt}\vec{r_i}^{O}$$

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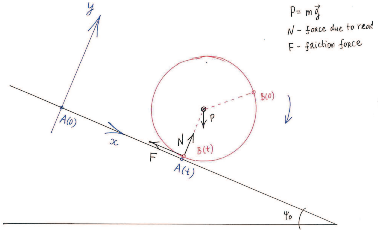
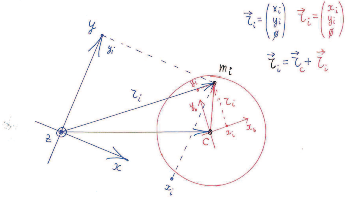
Example: disc rolling on inclined line

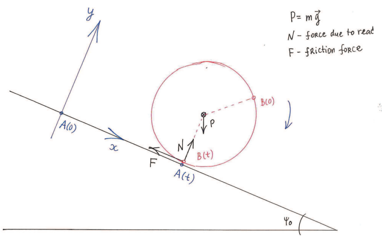


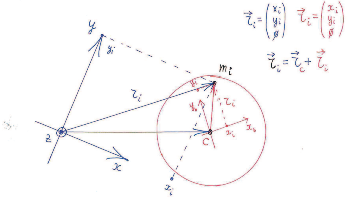


$$\vec{r}_i^{\,O} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta}\vec{r}_i^{\,D}, \quad \vec{v}_i = \frac{d}{dt}\vec{R}_C + \frac{d}{dt}\vec{r}_i^{\,O}$$

$$\begin{aligned} \vec{L}^C &= \sum_i \vec{r}_i^{\,O} \times m_i \vec{v}_i = \sum_i R_{z,\theta}\vec{r}_i^{\,D} \times m_i \left(\frac{d}{dt}\vec{R}_C + \frac{d}{dt}\left[R_{z,\theta}\vec{r}_i^{\,D} \right] \right) \\ &= \left[0;\, 0;\, J_z \frac{d}{dt}\theta \right] \end{aligned}$$

Example: disc rolling on inclined line

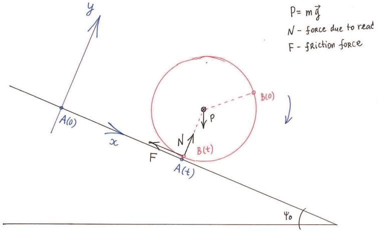


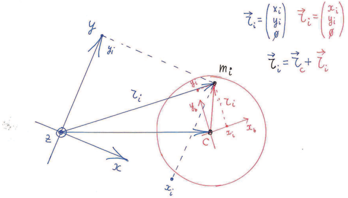


$$\vec{r_i}^{O} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 0 \end{bmatrix} = R_{z,\theta}\vec{r_i}^{D}, \quad \vec{v_i} = \frac{d}{dt}\vec{R}_C + \frac{d}{dt}\vec{r_i}^{O}$$

$$\vec{L}^{C} = \sum_i \vec{r_i}^{O} \times m_i \vec{v_i} = \sum_i R_{z,\theta}\vec{r_i}^{D} \times m_i \left(\frac{d}{dt}\vec{R}_C + \frac{d}{dt}\left[R_{z,\theta}\vec{r_i}^{D} \right] \right)$$

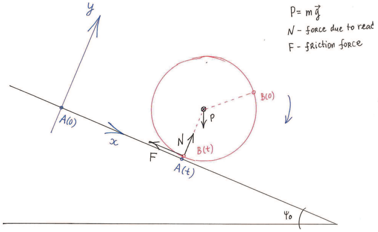
$$= \left[0;\ 0;\ J_z \frac{d}{dt}\theta \right], \qquad \text{for a disc } J_z = \frac{1}{2}(\text{Mass}) * (\text{Radius})^2$$

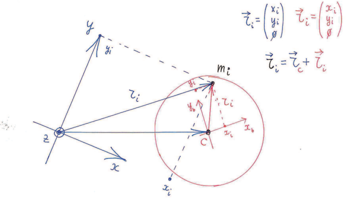




The rate of change of angular momentum $\vec{L}^C$ is then

$$\frac{d}{dt}\begin{bmatrix} 0 \\ 0 \\ J_z\dot{\theta} \end{bmatrix} = \vec{r}_P{}^O \times \vec{P} + \vec{r}_N{}^O \times \vec{N} + \vec{r}_F{}^O \times \vec{F}$$





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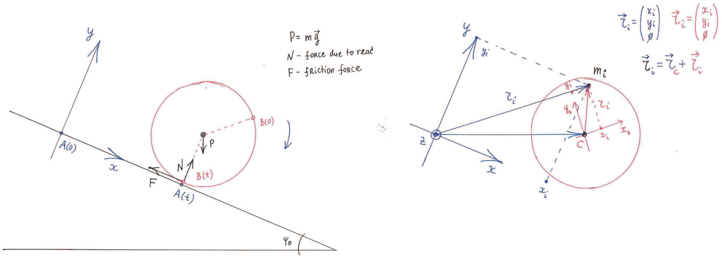
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \vec{P} + \begin{bmatrix} 0 \\ -R_d \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ N_y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -R_d \\ 0 \end{bmatrix} \times \begin{bmatrix} -F_x \\ 0 \\ 0 \end{bmatrix}$$

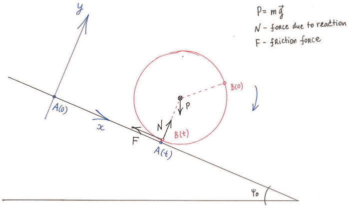
Example: disc rolling on inclined line



The rate of change of angular momentum $\vec{L}^C$ is then

$$\frac{d}{dt}\begin{bmatrix} 0 \\ 0 \\ J_z\dot{\theta} \end{bmatrix} = \vec{r_P}^O \times \vec{P} + \vec{r_N}^O \times \vec{N} + \vec{r_F}^O \times \vec{F} = \begin{bmatrix} 0 \\ 0 \\ -R_d \cdot F_x \end{bmatrix}$$

Example: disc rolling on inclined line

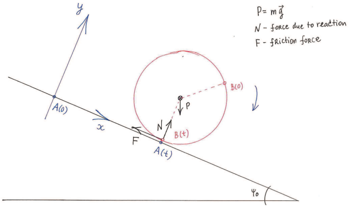


The dynamics of the system are

$$
\begin{aligned}
M \cdot \ddot{x}_C &= m \cdot g \cdot \sin \psi_0 - F \\
\ddot{y}_C &= 0 \\
0 &= N - m \cdot g \cdot \cos \psi_0 \\
J_z \cdot \ddot{\theta} &= -R_d \cdot F
\end{aligned}
$$

How many unknown variables?

Example: disc rolling on inclined line



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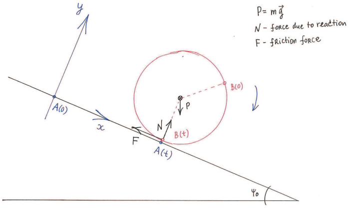

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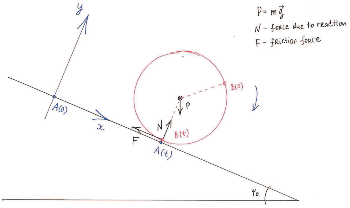
How many unknown variables? **Three**, they are: $x_C(\cdot)$, $\theta(\cdot)$, $F(\cdot)$



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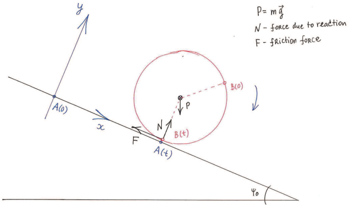
How many unknown variables? **Three**, they are: $x_C(\cdot)$, $\theta(\cdot)$, $F(\cdot)$

How many nontrivial equations?

Example: disc rolling on inclined line



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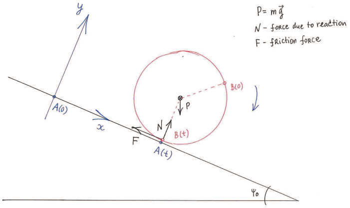
\end{aligned}
$$

How many unknown variables? **Three**, they are: $x_C(\cdot)$, $\theta(\cdot)$, $F(\cdot)$

How many nontrivial equations? **Two!**

Example: disc rolling on inclined line



The dynamics of the system are

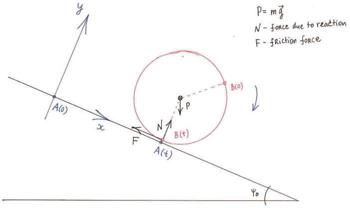
$$\begin{aligned} M \cdot \ddot{x}_C &= m \cdot g \cdot \sin\psi_0 - F \\ \ddot{y}_C &= 0 \\ 0 &= N - m \cdot g \cdot \cos\psi_0 \\ J_z \cdot \ddot{\theta} &= -R_d \cdot F \end{aligned}$$

How many unknown variables? **Three**, they are: $x_C(\cdot)$, $\theta(\cdot)$, $F(\cdot)$

How many nontrivial equations? **Two!**

We need one more equation for the system!

Example: disc rolling on inclined line



The dynamics of the system are

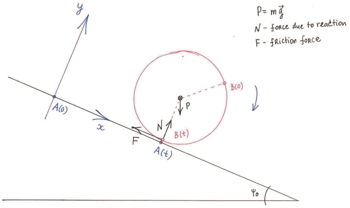
$$
\begin{aligned}
M \cdot \ddot{x}_C &= m \cdot g \cdot \sin\psi_0 - F \\
\ddot{y}_C &= 0 \\
0 &= N - m \cdot g \cdot \cos\psi_0 \\
J_z \cdot \ddot{\theta} &= -R_d \cdot F
\end{aligned}
$$

How many unknown variables? **Three**, they are: $x_C(\cdot)$, $\theta(\cdot)$, $F(\cdot)$

How many nontrivial equations? **Two!**

If the disc slides for some time interval $\Rightarrow$ $F(t) = \mu \cdot N$

Example: disc rolling on inclined line



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J_z \cdot \ddot{\theta} &= -R_d \cdot F
\end{aligned}
$$

How many unknown variables? **Three**, they are: $x_C(\cdot)$, $\theta(\cdot)$, $F(\cdot)$

How many nontrivial equations? **Two!**

If the disc slides for some time interval $\Rightarrow$ $F(t) = \mu \cdot N$

If the disc rolls without sliding for some interval

$$\Rightarrow \quad x_C(t) - x_C(0) = -R_d \cdot (\theta(t) - \theta(0))$$