

Lecture 7

Orbital Stability and Stabilization for Underactuated Systems

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Learning outcomes: Concepts of moving Poincaré sections, transverse coordinates and transverse linearization for a solution of a nonlinear system. Andronov-Vitt theorem. Choices of transverse coordinates and their linearization for motions of underactuated mechanical systems. Examples

1. Transverse dynamics and transverse coordinates

- Moving Poincare sections
- Andronov-Vitt theorem
- Challenges in orbital feedback stabilization
- Generic choice of transverse coordinates

2. Transverse COORDINATES for mechanical systems

3. Transverse LINEARIZATION for mechanical systems

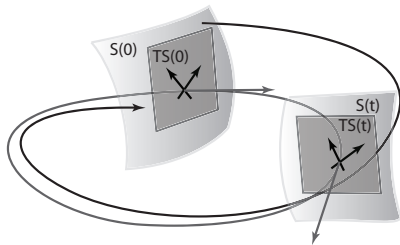
Transverse dynamics

Dynamics in a vicinity of a cycle

Given a T -periodic solution $x^*(\cdot)$,
 $x^*(t) = x^*(t + T) \forall t$, of the system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^{2n},$$

for analyzing its local properties
introduce a family of transverse
sections $\{S(\tau)\}$, $\tau \in [0, T]$, which
union can recreate a tubing vicinity of the cycle.

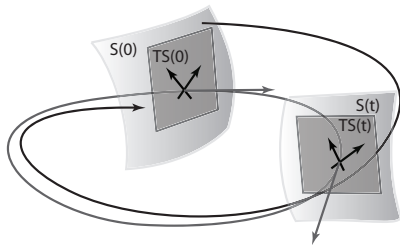


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Defining hypersurfaces $\{S(\tau)\}_{\tau \in [0, T]}$ implies a change of coordinates:

$$\mathbb{R}^{2n} \ni x(\tau) \mapsto [\theta(\tau) \in \mathbb{R}^1; x_{\perp} \in S(\tau)], \quad \tau \in [0, T]$$

such that in new coordinates the periodic solution $x^*(\cdot)$ is of the form

$$\theta = \theta^*(t), \quad \mathbf{x}_{\perp}^*(t) \equiv \mathbf{0}, \quad \forall t$$

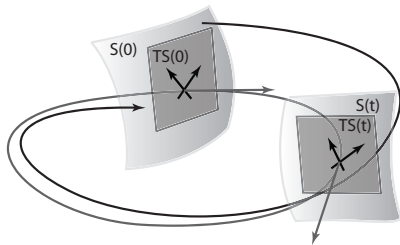
Orbital asymptotic stability of $x^*(\cdot)$ means that $x_{\perp}(t) \rightarrow 0$ as $t \rightarrow \infty$.

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The linearization of the full dynamics of the system in a vicinity of $x^*(\cdot)$

$$\dot{z} = \left[\frac{\partial}{\partial x} f(x) \right] \Big|_{x=x^*(t)} z = A(t)z, \quad z \in \mathbb{R}^{2n}$$

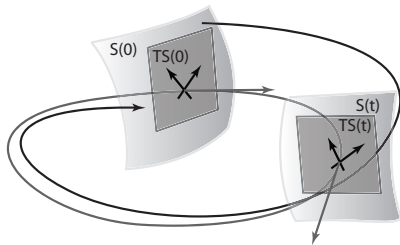
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- cannot be asymptotically stable since $\theta(t) \not\rightarrow 0$ as $t \rightarrow +\infty$
- has a linear invariant subspace of co-dimension 1, which vectors approximate time evolution of transverse coordinates $x_{\perp}(\cdot)$.

Andronov-Vitt theorem (1930)

Given a T -periodic solution

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Consider the $(2n \times 2n)$ -matrix function $\Phi(\cdot)$ defined as a solution of

$$\frac{d}{dt} \Phi(t) = A(t)\Phi(t), \quad \Phi(0) = I_{2n}$$

One of eigenvalues of the monodromy matrix $\Phi(T)$ equals to 1.

If the amplitudes of $(2n - 1)$ eigenvalues of the matrix $\Phi(T)$ are less than 1, then the solution $x^*(\cdot)$ is exponentially orbitally stable.

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Challenges in orbital feedback stabilization

Given a T -periodic solution $x^*(\cdot)$ of the nonlinear control system

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^{2n}, \quad u \in \mathbb{R}^m,$$

obtained in response of an input $u^*(\cdot) \equiv 0$, consider the task to design a controller for asymptotic orbital stabilization of $x^*(\cdot)$

Suppose it exists, then such orbitally stabilizing feedback controller

$$u = U(x)$$

will satisfy the interpolation condition

$$U(x)|_{x=x^*(t)} = u^*(t) \equiv 0, \quad \forall t$$

For a smooth $U(\cdot)$, it can be written as (see the Hadamard lemma)

$$u = K(x)x_{\perp}(x), \quad x_{\perp} \in \mathbb{R}^{2n-1}$$

where $x_{\perp}(\cdot)$ are transverse coordinates defined for the motion $x^*(\cdot)$

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The system dynamics in a vicinity of the solution $x^*(\cdot)$ can be written in new coordinates

$$\tilde{x} = [\theta; x_{\perp}], \quad \theta \in \mathbb{R}^1, \quad x_{\perp} \in \mathbb{R}^{2n-1}$$

as

$$\frac{d}{dt}\tilde{x} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})u.$$

With a feedback controller candidate

$$u = K(\tilde{x})x_{\perp}$$

the closed loop system becomes

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The linearization of the closed loop system in a vicinity of $x^*(\cdot)$ is

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By Andronov-Vitt theorem the linear feedback controller with the gain $k(\cdot)$ is not stabilizing the origin of this system

Challenges in orbital feedback stabilization

Orbital stabilization of the solution $x^*(\cdot)$ the system **by linearization**

$$x \in \mathbb{R}^{2n} : \quad \frac{d}{dt}x = f(x) + g(x)u, \quad [x^*(\cdot); u^*(\cdot)]$$

requires stabilization of **linearization** of transverse coordinates dynamics

This means that with change of coordinates

$$x \mapsto \tilde{x} = [\theta \in \mathbb{R}^1; x_{\perp} \in \mathbb{R}^{2n-1}]$$

it requires stabilizing an **invariant subspace** of the linearization

$$\dot{z} = A(t)z + B(t)v, \quad v = k(t)\delta x_{\perp}, \quad z \in \mathbb{R}^{2n}, \delta x_{\perp} \in \mathbb{R}^{2n-1}, v \in \mathbb{R}^n$$

of the system defined by variations of transverse coordinates $x_{\perp}(\cdot)$:

$$\delta x_{1\perp} \equiv 0, \quad \delta x_{2\perp} \equiv 0, \quad \dots, \quad \delta x_{(2n-1)\perp} \equiv 0$$

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Generic choice of transverse coordinates

Given a T -periodic pair $[x^*(\cdot), u^*(\cdot)]$ of the system

$$\frac{d}{dt}x = f(x) + g(x)u, \quad x \in \mathbb{R}^{2n}, \quad u \in \mathbb{R}^m$$

how to define the transverse coordinates $x_\perp \in \mathbb{R}^{2n-1}$ for the motion?

Step 1: Given $t_\bullet \in [0, T]$ and the vector $v_0 := \frac{d}{dt} x^*(t_\bullet)$, choose a set of vectors

$$v_1 \in \mathbb{R}^{2n}, \quad v_2 \in \mathbb{R}^{2n}, \quad \dots, \quad v_{2n-1} \in \mathbb{R}^{2n}$$

such that the $(2n \times 2n)$ -matrix

$$[v_0, v_1, v_2, \dots, v_{2n-1}]$$

is of rank $2n$

Step 2: Choose $v_i(t)$ for all $t \in [0, T]$ to be periodic and smooth.

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Step 1: Given $t_\bullet \in [0, T]$ and the vector $v_0 := \frac{d}{dt}x^*(t_\bullet)$, choose a set of vectors

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such that the $(2n \times 2n)$ -matrix

$$[v_0, v_1, v_2, \dots, v_{2n-1}]$$

is of rank $2n$

Step 2: Choose $v_i(t)$ for all $t \in [0, T]$ to be periodic and smooth.

Transverse coordinates

Alternative Representations of a Motion

Given a motion $q^*(t)$, $t \in [0, T]$ of a mechanical underactuated system

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}} \right] - \frac{\partial \mathcal{L}}{\partial q} = B(q)u, \quad q \in \mathbb{R}^n, \quad u \in \mathbb{R}^{n-1}$$

obtained in response to the input signal $u^*(t)$, $t \in [0, T]$

if the motion admits kinematic relations

$$q_1 = \phi_1(\theta), \quad q_2 = \phi_2(\theta), \quad \dots, \quad q_n = \phi_n(\theta)$$

valid between the coordinates q_i and a new scalar variable θ

then there is a scalar system with coefficients defined by $\mathcal{L}$ and $\{\phi_i\}_{i=1}^n$

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0 \quad \text{and its solution } \theta^*(\cdot)$$

which allows re-writing the motion $q^*(\cdot)$ on the time interval $[0, T]$ as

$$q_1^*(t) = \phi_1(\theta^*(t)), \quad q_2^*(t) = \phi_2(\theta^*(t)), \quad \dots, \quad q_n^*(t) = \phi_n(\theta^*(t))$$

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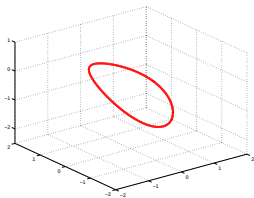
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Transverse Coordinates for a Motion :



Given a T -periodic motion of n -DOF MS

$$q^*(t) = [q_1^*(t); q_2^*(t); \dots; q_n^*(t)] = q^*(t+T),$$

a scalar θ , the functions for its representation

$$\phi_1(\cdot), \quad \phi_2(\cdot), \quad \dots, \quad \phi_n(\cdot)$$

The $(n+1)$ coordinates for describing the dynamics in a vicinity of $q^*(t)$

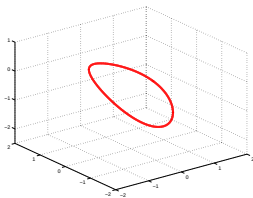
$$\theta, \quad y_1 := q_1 - \phi_1(\theta), \quad \dots, \quad y_n := q_n - \phi_n(\theta)$$

are excessive! And, therefore, they are not independent and one variable is redundant!

Then one of candidates for transverse coordinates $x_\perp$ for a motion of the system with $x = (q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)^T$ is given by $(2n-1)$ -quantities

$$x_\perp = [I(\theta, \dot{\theta}, \theta^*(0), \dot{\theta}^*(0)); y_1; \dots; y_{n-1}; \dot{y}_1; \dots; \dot{y}_{n-1}]$$

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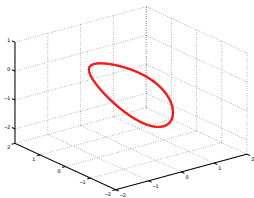
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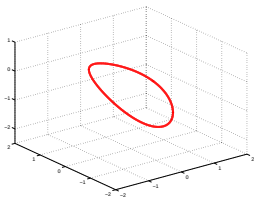
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Transverse linearization

Steps in Computing Transverse Linearization

Given a T -periodic motion $q^*(t) = (q_1^*(t), q_2^*(t), \dots, q_n^*(t))^T$ of the mechanical system

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}} \right] - \frac{\partial \mathcal{L}}{\partial q} = B(q)u$$

Step 1: Rewrite the dynamics in the coordinates: θ, y with

$$y_1 = q_1 - \phi_1(\theta), \quad \dots, \quad y_{n-1} = q_{n-1} - \phi_{n-1}(\theta)$$

Step 2: Define a feedback transformation

$$u = (u_1, \dots, u_{n-1}) \rightarrow v = (v_1, \dots, v_{n-1})$$

such that the dynamics of y become linear

$$\ddot{y} = v, \quad \ddot{\theta} = N(\theta, \dot{\theta}, y, \dot{y}, \ddot{y})$$

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Steps in Computing Transverse Linearization: θ -dynamics

Since the θ -dynamics with constraints $y = \dot{y} = \ddot{y} = \mathbf{v} = 0$

$$\ddot{\theta} = N(\theta, \dot{\theta}, y, \dot{y}, \ddot{y})$$

become

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

Therefore, the dynamics of θ -variable can be rewritten as

$$\begin{aligned}\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) &= g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})y + \\ &\quad + g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\dot{y} + g_v(\theta, \dot{\theta}, y, \dot{y})\ddot{y}\end{aligned}$$

The functions $g_y(\cdot)$, $g_{\dot{y}}(\cdot)$ and $g_v(\cdot)$ are to be found based on

- the Lagrangian $\mathcal{L}(\cdot)$ of the system;
- the functions $\phi_1(\cdot), \dots, \phi_{n-1}(\cdot)$ representing the motion

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Computing Transverse Linearization: Passivity Relation for $I(\cdot)$

The mechanical system written in y, θ -coordinates is

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We are searching for linearization of transverse dynamics

$$\mathbf{x}_{\perp} = \left[I(\theta, \dot{\theta}, \theta^*(0), \dot{\theta}^*(0)); y; \dot{y} \right]$$

around the target solution: $\theta(t) = \theta^*(t)$, $y(t) \equiv 0$

To do so, one can use the passivity relation for $\frac{d}{dt}I(\cdot)$ found previously

$$\begin{aligned}\frac{d}{dt}I(\theta, \dot{\theta}, \mathbf{a}, \mathbf{b}) &\equiv \dot{\theta} \left\{ \frac{2}{\alpha(\theta)} W - \frac{2\beta(\theta)}{\alpha(\theta)} I(\theta, \dot{\theta}, \mathbf{a}, \mathbf{b}) \right\} \\ &\equiv \dot{\theta} \left\{ \frac{2}{\alpha(\theta)} \left[g_y(\cdot)y + g_{\dot{y}}(\cdot)\dot{y} + g_v(\cdot)\mathbf{v} \right] - \frac{2\beta(\theta)}{\alpha(\theta)} I(\theta, \dot{\theta}, \mathbf{a}, \mathbf{b}) \right\}\end{aligned}$$

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Transverse Linearization: the Final Form

The linear control system with $(2n - 1)$ states and $(n - 1)$ control inputs

$$\begin{aligned}\frac{d}{dt}l_{\heartsuit}(t) &= \frac{2\dot{\theta}^*(t)}{\alpha(\theta^*(t))} \left\{ \left[\tilde{g}_y(t)y_{\heartsuit}(t) + \tilde{g}_{\dot{y}}(t)\dot{y}_{\heartsuit}(t) + \tilde{g}_v(t)v_{\heartsuit}(t) \right] - \right. \\ &\quad \left. - \beta(\theta^*(t))l_{\heartsuit}(t) \right\} \\ \ddot{y}_{\heartsuit}(t) &= v_{\heartsuit}(t)\end{aligned}$$

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with the functions

$$\begin{aligned}\tilde{g}_y(t) &= g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y}) \Big|_{\substack{\theta = \theta^*(t), \dot{\theta} = \dot{\theta}^*(t) \\ \ddot{\theta} = \ddot{\theta}^*(t), y = \dot{y} = 0}} \\ \tilde{g}_{\dot{y}}(t) &= g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y}) \Big|_{\substack{\theta = \theta^*(t), \dot{\theta} = \dot{\theta}^*(t) \\ \ddot{\theta} = \ddot{\theta}^*(t), y = \dot{y} = 0}} \\ \tilde{g}_v(t) &= g_v(\theta, \dot{\theta}, y, \dot{y}) \Big|_{\substack{\theta = \theta^*(t), \dot{\theta} = \dot{\theta}^*(t) \\ y = \dot{y} = 0}}\end{aligned}$$

**Transverse Linearization of the Mechanical System
in a Vicinity of the Motion $q^*(\cdot)$**

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Any **linear** feedback controller of the form

$$v_{\heartsuit}(t) = K(t) \begin{bmatrix} l_{\heartsuit} \\ y_{\heartsuit} \\ \dot{y}_{\heartsuit} \end{bmatrix}, \quad K(t) = K(t + T),$$

that stabilizes the origin of the linear control system can be transformed into a **nonlinear controller that stabilizes orbitally** the nominal periodic motion $q^*(\cdot)$ of the nonlinear underactuated mechanical system