Lecture 4

Introduction to Stability of Mechanical Systems

Anton Shiriaev
Norwegian University of Science and Technology
Trondheim, Norway

Learning outcomes: Tools for analysis of stability of a motion of a nonlinear mechanical system. Examples.

Outline

1. Example: a mathematical pendulum
2. Example: a pendulum on a cart
3. Lagrange-Dirichlet theorem
4. Example: restricted 3 body problem

Example: a mathematical pendulum

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Let us investigate a stability of equilibriums of the system

$$\ddot{\theta} + \frac{g}{l} \cdot \sin\theta = 0. \tag{1}$$

There are two equilibriums $(\bmod 2\pi)$: $\theta_e = 0$ and $\theta_e = \pi$.

Linearization of the dynamics at $\theta_e = \pi$ results in

$$\ddot{z} - \frac{g}{l} \cdot z = 0.$$

The equilibrium of this linear system is unstable. Therefore, the equilibrium $\theta_e = \pi$ of the nonlinear system (1) is unstable as well.

Linearization of the dynamics at $\theta_e = 0$ results in

$$\ddot{z} + \frac{g}{l} \cdot z = 0.$$

This linear system has the center at the origin and the nonlinear dynamics (1) has the first integral. Therefore, the nonlinear system (1) has the center at $\theta_e = 0$ and this equilibrium is stable!

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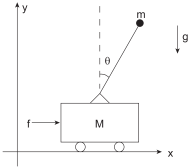
Consider a pendulum (a point of a mass m at the distance l from the suspension point) attached to a cart of a mass M, which freely moves on the horizontal with $f = 0$.



When $M = m = l = 1$ the dynamics in coordinates (x, θ) are

$$\begin{aligned} 2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 &= 0 \ (= f) \\ \cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta &= 0 \end{aligned}$$

and have two sets of equlibriums:

- the pendulum is up, the cart is in any position: $\theta_e = 0$, $\forall\, x_e$
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Let us investigate their stability!

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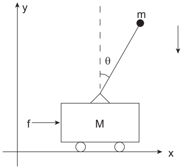

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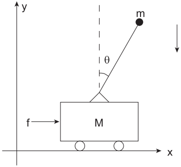

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The system
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has two integrals of motion (CoM): the total energy

$$E = \frac{1}{2}\begin{bmatrix}\dot{x} \\ \dot{\theta}\end{bmatrix}^T \begin{bmatrix} 2 & \cos\theta \\ \cos\theta & 1 \end{bmatrix} \begin{bmatrix}\dot{x} \\ \dot{\theta}\end{bmatrix} + g\cdot\cos\theta$$

and the momentum conjugated to the x-coordinate

$$P(x(t),\theta(t),\dot{x}(t),\dot{\theta}(t)) = 2\cdot\dot{x}(t) + \cos\theta(t)\cdot\dot{\theta}(t)$$

The last relation can be integrated again and leads to

$$\begin{aligned} x(t) &= x(0) + \tfrac{1}{2}\sin\theta(0) - \tfrac{1}{2}\sin\theta(t) + \tfrac{1}{2}P(x(0),\theta(0),\dot{x}(0),\dot{\theta}(0))\cdot t \\ &= \boxed{-\tfrac{1}{2}\sin\theta(t) + C_0 + C_1\cdot t} \end{aligned}$$

If one is able to compute $\theta(\cdot)$, then the formula gives $x(\cdot)$!

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For decoupling dynamics of the θ-variable in the system

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one can use the second equation.

Indeed

$$\cos\theta\cdot\overbrace{\tfrac{1}{2}\left(\sin\theta\cdot\dot{\theta}^2-\cos\theta\cdot\ddot{\theta}\right)}^{=\ddot{x}}+\ddot{\theta}-g\cdot\sin\theta=0$$

substituting and collecting the terms one derives the equation

$$\boxed{\left(1-\tfrac{1}{2}\cos^2\theta\right)\cdot\ddot{\theta}+\tfrac{1}{2}\cos\theta\cdot\sin\theta\cdot\dot{\theta}^2-g\cdot\sin\theta=0}$$

What can one say on the dynamics of this system
in a vicinity of its equilibriums at 0 and at π?

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Its linearization at $\theta_e = \pi$ is stable

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In addition the system has the first integral

$$E_{red}=\tfrac{1}{2}\left(1-\tfrac{1}{2}\cos^2\theta\right)\cdot\dot{\theta}^2+g\cdot\cos\theta$$

The nonlinear system has the center at $\theta_e = \pi$!

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To conclude:

- Any cart pendulum solution can be written as

$$\boxed{x(t) = \frac{1}{2}\sin\theta(t) + C_0 + C_1 t}$$

with $\theta(t)$ being a solution of reduced dynamics.

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To conclude:

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- In a vicinity of the upright equilibrium $[\theta_e = 0,\, x = x_e]$ the reduced dynamics is hyperbolic, therefore **any of upright equilibriums is unstable**.

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- In a vicinity of the upright equilibrium $[\theta_e = 0,\, x = x_e]$ the reduced dynamics is hyperbolic, therefore **any of upright equilibriums is unstable**.

- In a vicinity of the downward equilibrium $[\theta_e = \pi,\, x = x_e]$ the reduced dynamics is stable, but $x(t)$ will drift with $C_1 \neq 0$. Hence **any of downward equilibriums is unstable as well**.

Lagrange-Dirichlet theorem

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Theorem (1788)

If at the position of an isolated equilibrium of a conservative mechanical system with holonomic constraints the potential energy Π has a strict minimum, then this equilibrium is stable.

Example: restricted 3 body problem

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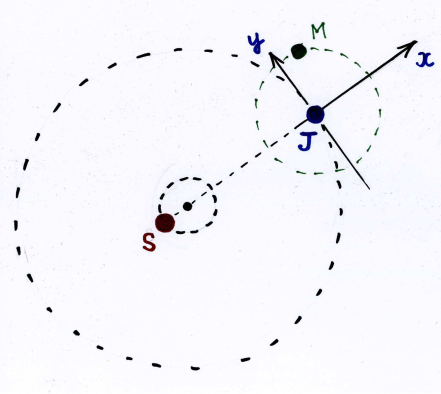
Equations of motion for the position of the Moon in rotating coordinate frame are

$$\begin{cases} \ddot{x} - 2m\dot{y} & = & \frac{\partial}{\partial x} F \\ \ddot{y} + 2m\dot{x} & = & \frac{\partial}{\partial y} F \end{cases}$$

Here

$$F = \frac{\kappa}{\sqrt{x^2 + y^2}} + \tfrac{3}{2} m^2 x^2$$

m, κ are positive constants.



The system has the invariant: $I = \dot{x}^2 + \dot{y}^2 - 2F(x, y) + C$

Task: Analyze the dynamics in a vicinity of the periodic motion

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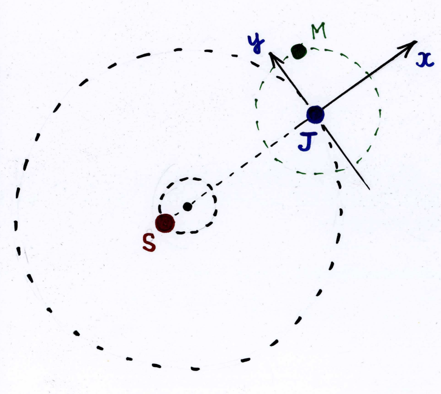

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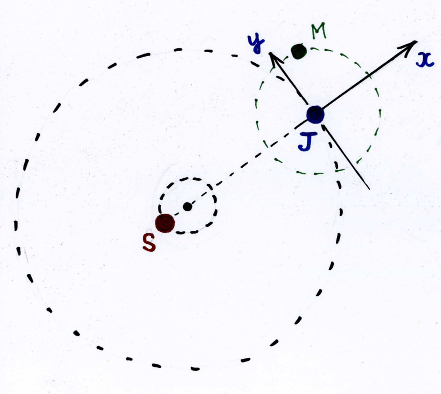

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Example: elements of theory of G.W. Hill

Denote as $[\,x_p(t),\, y_p(t)\,]$ the periodic solution

Perturbed solutions $[x_p(t) + \delta x(t),\ y_p(t) + \delta y(t)]$ defined by

$$\frac{d^2}{dt^2}\left[\delta x\right] - 2m\frac{d}{dt}\left[\delta y\right] = $$

$$= \left[\frac{\partial^2}{\partial x^2}F(x_p(t), y_p(t))\right]\delta x + \left[\frac{\partial^2}{\partial x \partial y}F(x_p(t), y_p(t))\right]\delta y$$

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The integral Jacobi $I(\cdot)$ gives another relation

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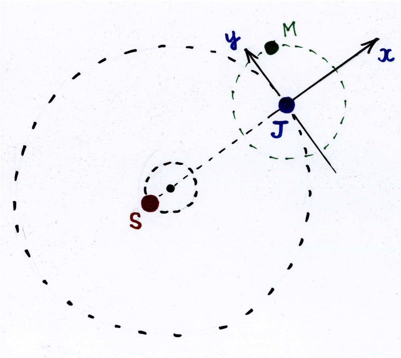

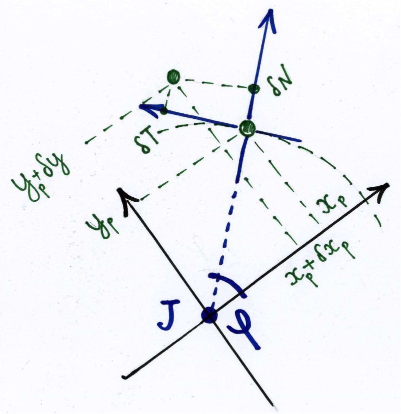

Transform of coordinates into normal (δN) and tangent (δT)

$$\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \delta T \\ \delta N \end{bmatrix}$$

Example: elements of theory of G.W. Hill

In a vicinity of the motion the original coordinates

$$\left[x,\, y,\, \dot{x},\, \dot{y}\right]$$

are changed into

$$\left[\phi,\, I,\, N,\, \dot{N}\right]$$

The linearization of $\phi(\cdot)$ is not important: it perpetually rotates

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The linearization of $I(\cdot)$ is straightforward: $\frac{d}{dt}\left[\delta I\right] = 0$

The linearization of $[N, \dot{N}]$ is the famous Hill's equation

$$\frac{d^2}{dt^2}\left[\delta N\right] + \Phi(t)\delta N = 0$$

Example: elements of theory of G.W. Hill

In a vicinity of the motion the original coordinates

$$\left[x,\, y,\, \dot{x},\, \dot{y}\right]$$

are changed into

$$\left[\phi,\, I,\, N,\, \dot{N}\right]$$

The linearization of $\phi(\cdot)$ is not important: it perpetually rotates

The linearization of $I(\cdot)$ is straightforward: $\frac{d}{dt}\left[\delta I\right] = 0$

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Outcome of the example

Analysis of dynamics in a vicinity of the motion's orbit requires:

- Decomposition of coordinates into
  - transverse to the trajectory (dim $= 2n - 1$)
  - along the trajectory (dim $= 1$)

  In the example they are

$$\left[I,\ N,\ \dot{N}\right] \quad \text{and} \quad \phi$$

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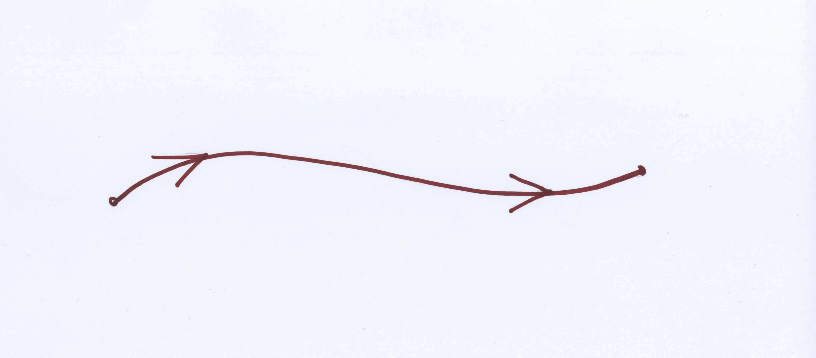
- Presence of invariants allows to reduce a number of transverse coordinates with non-trivial dynamics.

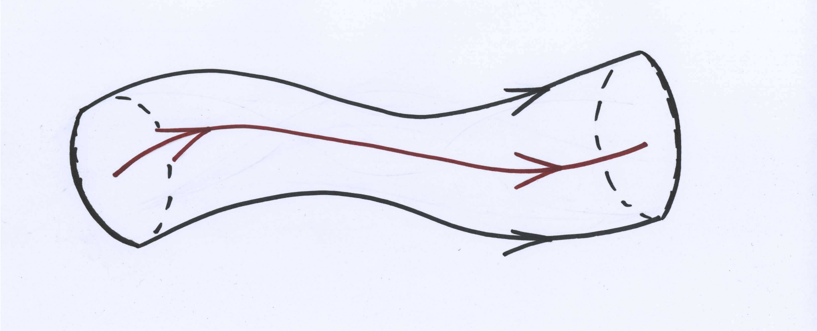
  In the example the integral Jacobi $I(\cdot)$ is excluded.

Outcome of the example: geometrical interpretation



Given a trajectory of a nominal motion

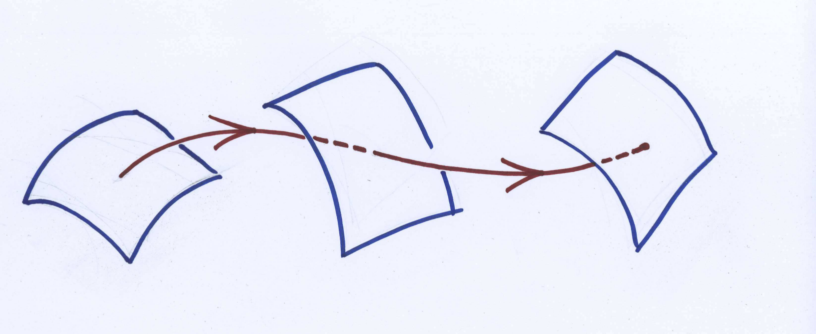
Outcome of the example: geometrical interpretation



We would like to analyze properties of the dynamics

in its tubing vicinity

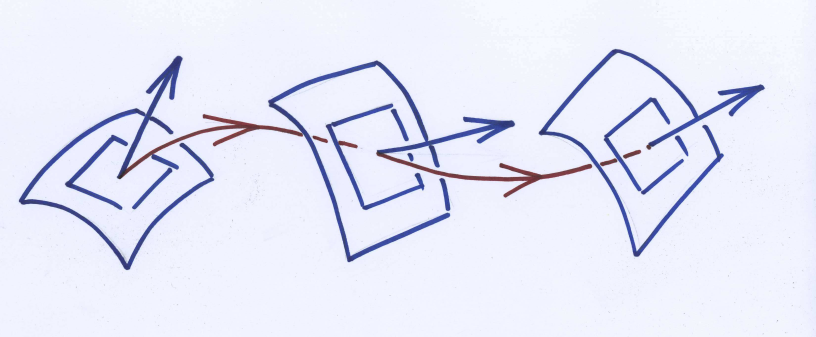
Outcome of the example: geometrical interpretation



Introduce a family of dis-joint transverse surfaces

that are continuously slicing this vicinity

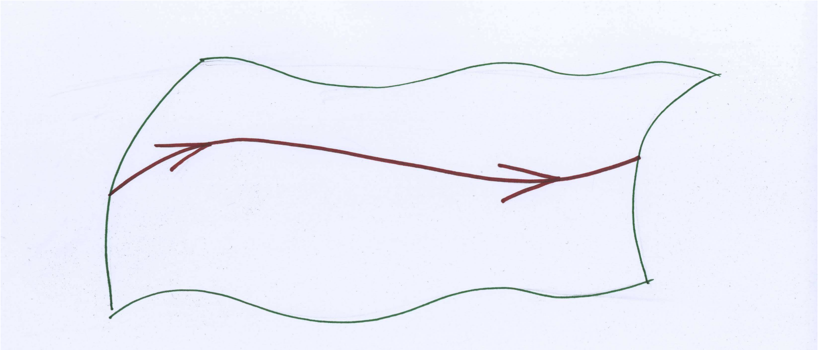
Outcome of the example: geometrical interpretation



For the linearization of the dynamics the surfaces

are substituted by tangent planes

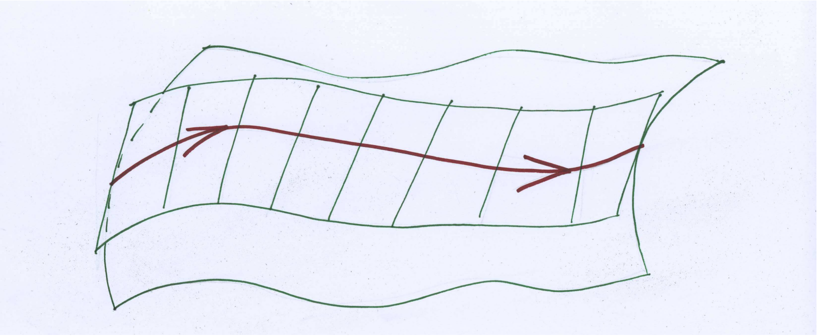
Outcome of the example: geometrical interpretation



If the dynamics have some invariants,

then they define a manifold

Outcome of the example: geometrical interpretation



For the linearization we consider the linear subspaces

that are tangent to to the trajectory along this manifold

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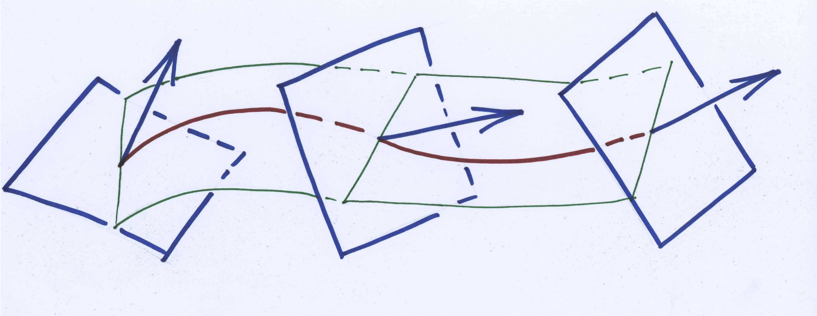
Outcome of the example: geometrical interpretation



Evolution of coordinates on these linear subspaces will define

linearization of transverse coordinates with nontrivial behavior