Lecture 5

Transverse Linearization for Nonlinear Systems

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Learning outcomes: Concepts of moving Ponicare sections, transverse coordinates and transverse linearization for a solution of a nonlinear system. Andronov-Vitt theorem. Zhukovsky stability and Leonov lemma. Examples

Outline

1. Transverse dynamics and transverse coordinates
   - Moving Poincare sections
   - Andronov-Vitt theorem
   - Challenges in orbital feedback stabilization
   - Generic choice of transverse coordinates
2. Zhukovsky stability and linearization
   - Definition
   - Leonov lemma
   - Example: Energy-based control of a pendulum

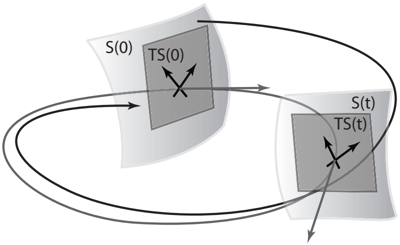
Transverse dynamics

Dynamics in a vicinity of a cycle

Given a T-periodic solution $x^*(\cdot)$, $x^*(t) = x^*(t+T)\ \forall\, t$, of the system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^{2n},$$

for analyzing its local properties introduce a family of transverse sections $\{S(\tau)\}$, $\tau \in [0, T]$, which union can recreate a tubing vicinity of the cycle.



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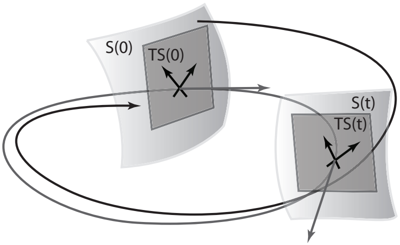

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Defining hypersurfaces $\{S(\tau)\}_{\tau \in [0,T]}$ implies a change of coordinates:

$$\mathbb{R}^{2n} \ni x(\tau) \mapsto \left[\theta(\tau) \in \mathbb{R}^1;\, x_\perp \in S(\tau)\right], \qquad \tau \in [0, T]$$

such that in new coordinates the periodic solution $x^*(\cdot)$ is of the form

$$\theta = \theta^*(t), \quad x_\perp^*(t) \equiv \mathbf{0}, \qquad \forall\, t$$

Orbital asymptotic stability of $x^*(\cdot)$ means that $x_\perp(t) \to 0$ as $t \to \infty$.

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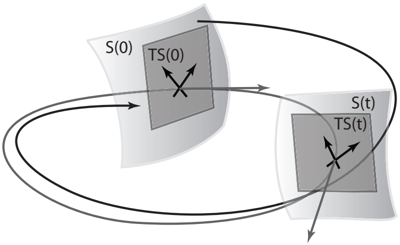

The linearization of the full dynamics of the system in a vicinity of $x^*(\cdot)$

$$\dot{z} = \left[\frac{\partial}{\partial x} f(x)\right]\Big|_{x=x^*(t)} z = A(t)z, \quad z \in \mathbb{R}^{2n}$$

- cannot be asymptotically stable since $\theta(t) \not\to 0$ as $t \to +\infty$

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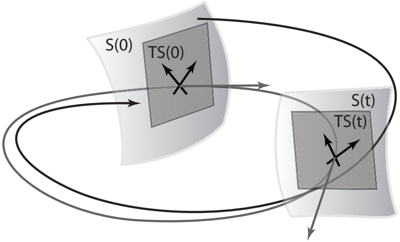

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- cannot be asymptotically stable since $\theta(t) \not\to 0$ as $t \to +\infty$
- has a linear invariant subspace of co-dimension 1, which vectors approximate time evolution of transverse coordinates $x_\perp(\cdot)$.

Andronov-Vitt theorem (1930)

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Consider the $(2n \times 2n)$-matrix function $\Phi(\cdot)$ defined as a solution of

$$\tfrac{d}{dt}\Phi(t) = A(t)\Phi(t), \qquad \Phi(0) = I_{2n}$$

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If the amplitudes of $(2n-1)$ eigenvalues of the matrix $\Phi(T)$ are less than 1, then the solution $x^*(\cdot)$ is exponentially orbitally stable.

Challenges in orbital feedback stabilization

Given a T-periodic solution $x^*(\cdot)$ of the nonlinear control system

$$\dot{x} = f(x) + g(x)u, \qquad x \in \mathbb{R}^{2n},\ u \in \mathbb{R}^m,$$

obtained in response of an input $u^*(\cdot) \equiv 0$, consider the task to design a controller for asymptotic orbital stabilization of $x^*(\cdot)$

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Suppose it exists, then such orbitally stabilizing feedback controller

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For a smooth $U(\cdot)$, it can be written as (see Hadamard lemma)

$$u = K(x)x_\perp(x), \qquad x_\perp \in \mathbb{R}^{2n-1}$$

where $x_\perp(\cdot)$ are transverse coordinates defined for the motion $x^*(\cdot)$

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The system dynamics in a vicinity of the solution $x^*(\cdot)$ can be written in new coordinates

$$\tilde{x} = [\theta;\, x_\perp], \quad \theta \in \mathbb{R}^1,\ x_\perp \in \mathbb{R}^{2n-1}$$

as

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With a feedback controller candidate

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By Andronov-Vitt theorem the linear feedback controller with the gain $k(\cdot)$ is not stabilizing the origin of this system

Challenges in orbital feedback stabilization

Orbital stabilization of the solution $x^*(\cdot)$ the system **by linearization**

$$x \in \mathbb{R}^{2n} : \quad \tfrac{d}{dt}x = f(x) + g(x)u, \quad [x^*(\cdot);\ u^*(\cdot)]$$

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This means that with change of coordinates

$$x \mapsto \tilde{x} = [\theta \in \mathbb{R}^1;\ x_\perp \in \mathbb{R}^{2n-1}]$$

it requires stabilizing an **invariant subspace** of the linearization

$$\dot{z} = A(t)z + B(t)v, \quad v = k(t)\delta x_\perp, \quad z \in \mathbb{R}^{2n},\ \delta x_\perp \in \mathbb{R}^{2n-1},\ v \in \mathbb{R}^n$$

of the system defined by variations of transverse coordinates $x_\perp(\cdot)$:

$$\delta x_{1\perp} \equiv 0, \quad \delta x_{2\perp} \equiv 0, \quad \ldots, \quad \delta x_{(2n-1)\perp} \equiv 0$$

Generic choice of transverse coordinates

Given a T-periodic pair $[\,x^*(\cdot), u^*(\cdot)\,]$ of the system

$$\frac{d}{dt}x = f(x) + g(x)u, \quad x \in \mathbb{R}^{2n},\ u \in \mathbb{R}^m$$

how to define the transverse coordinates $x_\perp \in \mathbb{R}^{2n-1}$ for the motion?

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Step 1: Given $t_\bullet \in [0, T]$ and the vector $v_0 := \frac{d}{dt} x^*(t_\bullet)$, choose a set of vectors

$$v_1 \in \mathbb{R}^{2n}, \quad v_2 \in \mathbb{R}^{2n}, \quad \ldots, \quad v_{2n-1} \in \mathbb{R}^{2n}$$

such that the $(2n \times 2n)$-matrix

$$\left[v_0, \, v_1, \, v_2, \, \ldots, \, v_{2n-1} \right]$$

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Step 2: Choose $v_i(t)$ for all $t \in [0, T]$ to be periodic and smooth.

Zhukovsky stability and linearization

Zhukovsky stability

Consider a nonlinear dynamic system, one of its solutions

$$\tfrac{d}{dt}x = f(x), \quad x^0(t) = x(t, x_0) \in \mathbb{R}^{2n}, \; t \in [0, +\infty)$$

and a set of homeomorphisms (re-parametrizations of time)

$$Hom := \{\tau(\cdot) | \, \tau : [0, +\infty) \to [0, +\infty), \, \tau(0) = 0\}$$

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Definition

The solution $x^0(t) = x(t, x_0)$ of the system is referred to as

- **stable**, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y_0 \in \mathbb{R}^n$ from the set $|x_0 - y_0| < \delta$ there exists $\tau(\cdot) \in Hom$ such that
$$|x(t, x_0) - x(\tau(t), y_0)| < \varepsilon \quad \forall \, t \geq 0$$

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$$|x(t, x_0) - x(\tau(t), y_0)| < \varepsilon \quad \forall\, t \geq 0$$
- **asymptotically stable**, if, in addition, for some $\delta_0 > 0$ and any y_0 from the set $\{y_0 : |x_0 - y_0| < \delta_0\}$ there exists $\tau(\cdot) \in Hom$ such that
$$\lim_{t \to +\infty} |x(t, x_0) - x(\tau(t), y_0)| = 0$$

Lyapunov, Zhukovsky and Poincare stability

Consider a nonlinear dynamic system and one of its solutions

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$\boxed{x^0(t) \text{ is stable in a sense of Poincare}}$

Linearization

Linearization of a nonlinear dynamic system around its solution

$$\frac{d}{dt}x = f(x), \quad x^0(t) = x(t, x_0) \in \mathbb{R}^{2n},\ t \in [0, +\infty)$$

is the method for approximating the error by solving a linear system

$$x(t, y_0) - x(t, x_0) \approx w(t) \quad \Leftarrow \frac{d}{dt}w = A(t)w,\ w(0) = y_0 - x_0$$

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How to compute the first order approximation for the error

$$x(\tau(t), y_0) - x(t, x_0) \approx v(t) \quad \boxed{\Leftarrow \tfrac{d}{dt}v = \mathcal{A}(t)v,\ v(0) = y_0 - x_0?}$$

Leonov lemma

Given a nominal $x(t, x_0)$ and a perturbed $x(t, y_0)$ solutions of the system

$$\frac{d}{dt}x = f(x), \quad x \in \mathbb{R}^{2n},$$

suppose there is a homeomorphism $\tau = \tau(t) \in \mathit{Hom}$ defined by the Eqn

$$[x(\tau, y_0) - x(t, x_0)]^T f(x(t, x_0)) \equiv 0, \quad \forall\, t \geq 0$$

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Consider the difference between these solutions at times t and $t \to \tau(t)$

$$z(t) := x(\tau(t), y_0) - x(t, x_0), \quad z \in \mathbb{R}^{2n}$$

and re-write its time derivative as

$$\tfrac{d}{dt}z(t) \quad = \quad \tfrac{d}{d\tau}x(\tau(t), y_0)\,\tfrac{d}{dt}\tau(t) - \tfrac{d}{dt}x(t, x_0)$$

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$$\Rightarrow \quad \mathcal{A}(t) = \left\{ \frac{\partial f(x)}{\partial x} - \frac{f(x)f(x)^T}{|f(x)|^2}\left(\frac{\partial f(x)}{\partial x} + \left[\frac{\partial f(x)}{\partial x}\right]^T\right)\right\}\Bigg|_{x=x(t,x_0)}$$

Leonov lemma. Remarks

As seen, the matrix function $\mathcal{A}(\cdot)$ in the formula

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The vector $z(\cdot)$ has the same number of elements as the state $x(\cdot) \in \mathbb{R}^{2n}$

$$z(\cdot) = [z_1(\cdot);\, z_2(\cdot);\, \ldots;\, z_{2n}(\cdot)]\,,$$

and at each time moment satisfies the Eqn

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$\Rightarrow$ $z(\cdot)$ is an excessive set of transverse coordinates for $x(t, x_0)$

New auxiliary linear system!

Given a nominal $x(t, x_0)$ solution of the system

$$\frac{d}{dt}x = f(x), \quad x \in \mathbb{R}^{2n},$$

one can explore its (local) properties by analyzing linear systems:

$$\boxed{\frac{d}{dt}w = A(t)w} \quad \text{with} \quad A(t) := \frac{\partial}{\partial x}f(x)\Big|_{x=x^0(t)}$$

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Some properties of linearized dynamics can be linked to the corresponding properties of original nonlinear systems, but sometimes are NOT!

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Caution:

Some properties of linearized dynamics can be linked to the corresponding properties of original nonlinear systems, but sometimes are NOT!

Asymptotic stability of the origin of the linearized system **might not imply** asymptotic (Lyapunov, Zhukovsky, Poincare) stability of the corresponding solution of the nonlinear system!

New auxiliary linear system! (Cont'd)

Given a nominal $x^0(t) = x(t, x_0)$ solution of the system

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consider a solution $v(t) = v(t, v_0)$ of the linear system

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where the matrix function $\mathcal{A}(t)$ is defined in Leonov lemma.

Then the scalar function $\rho(\cdot)$ defined as a scalar product

$$\rho(t) := v(t)^T f(x^0(t))$$

is a constant on solutions of the linear system, i.e.

$$\rho(t) = v(t)^T f(x^0(t)) \equiv v(0)^T f(x^0(0)), \qquad \forall\, t \geq 0.$$

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Hence, if v_0 is chosen orthogonal to $f(x_0)$, then $v(t)^T f(x^0(t)) \equiv 0$, $\forall\, t$!

Example: Energy-based control of a pendulum

Consider the dynamics of a pendulum

$$a \cdot \ddot{\theta} + c \cdot \sin\theta = u, \quad a > 0,\ c > 0$$

augmented with the closed loop feedback controller

$$u = -\phi(\dot{\theta}) \cdot \Big[E(\theta, \dot{\theta}) - E_0\Big], \quad E(\theta, \dot{\theta}) = \tfrac{a}{2}\,\dot{\theta}^2 + c \cdot (1 - \cos\theta).$$

Here, E_0 is a non-negative constant, the scalar function $\phi(\cdot)$ is such that

$$\boxed{s \cdot \phi(s) > 0, \quad \forall\, s \neq 0}$$

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If $\theta_0(t)$ is a motion of the pendulum such that

$$E(\theta_0(t), \dot{\theta}_0(t)) = E_0 \quad \Rightarrow \quad u = 0.$$

Therefore, $\theta_0(t)$ is one of solutions of the closed loop system.

Let us investigate its stability

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$$\tfrac{d}{dt}\Big[E(\theta, \dot{\theta}) - E_0\Big]^2 \;=\; 2 \cdot \Big[E(\theta, \dot{\theta}) - E_0\Big] \cdot \tfrac{d}{dt} E(\theta, \dot{\theta})$$

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$$\begin{aligned} \tfrac{d}{dt}\Big[E(\theta, \dot{\theta}) - E_0\Big]^2 &= 2 \cdot \Big[E(\theta, \dot{\theta}) - E_0\Big] \cdot \tfrac{d}{dt} E(\theta, \dot{\theta}) \\ &= 2 \cdot \Big[E(\theta, \dot{\theta}) - E_0\Big] \cdot \Big\{\dot{\theta} \cdot u\Big\} \end{aligned}$$

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&= -2 \cdot \dot{\theta} \cdot \phi(\dot{\theta}) \cdot \Big[E(\theta,\dot{\theta}) - E_0\Big]^2
\end{aligned}$$

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&= -2 \cdot \dot{\theta} \cdot \phi(\dot{\theta}) \cdot \Big[E(\theta, \dot{\theta}) - E_0\Big]^2 \leq 0 \text{ !!!!}
\end{aligned}$$

Example (Cont'd)

The state space form $\dot{x} = f(x)$ of the closed loop system

$$a \cdot \ddot{\theta} + c \cdot \sin\theta = u = -\phi(\dot{\theta}) \cdot \left[E(\theta, \dot{\theta}) - E_0\right]$$

with $\{x_1 = \theta;\ x_2 = \dot{\theta}\}$ is written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{c}{a}\sin x_1 - \frac{\phi(x_2)}{a}\left[\frac{a}{2}x_2^2 + c(1-\cos x_1) - E_0\right] \end{bmatrix}$$

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The linear system defined for the solution $x_1^0(t) = \theta_0(t)$, $x_2^0(t) = \dot{\theta}_0(t)$ is

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathcal{A}(t)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

with

$$\mathcal{A}(t) = \left\{\frac{\partial f(x)}{\partial x} - \frac{f(x)f(x)^T}{|f(x)|^2}\left(\frac{\partial f(x)}{\partial x} + \left[\frac{\partial f(x)}{\partial x}\right]^T\right)\right\}\Bigg|_{x=x^0(t)}$$

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Solutions $v^{\perp}(\cdot)$ approximating transverse dynamics are defined by

$$0 \equiv v^{\perp}(t)^{T} f(x^0(t))$$

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$$0 \equiv v_1^{\perp}(t) f_1(x^0(t)) + v_2^{\perp}(t) f_2(x^0(t)) \ \Rightarrow\ v^{\perp}(t) = \lambda(t) \cdot \begin{bmatrix} -f_2(x^0(t)) \\ f_1(x^0(t)) \end{bmatrix}$$

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The linear system defined for the solution $x_1^0(t) = \theta_0(t)$, $x_2^0(t) = \dot{\theta}_0(t)$ is

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathcal{A}(t)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Solutions $v^{\perp}(\cdot)$ approximating transverse dynamics are defined by

$$0 \equiv v_1^{\perp}(t) f_1(x^0(t)) + v_2^{\perp}(t) f_2(x^0(t)) \;\Rightarrow\; v^{\perp}(t) = \lambda(t) \cdot \underbrace{\begin{bmatrix} -f_2(x^0(t)) \\ f_1(x^0(t)) \end{bmatrix}}_{=:\, v_0^{\perp}(t)}$$

Example (Cont'd)

The state space form $\dot{x} = f(x)$ of the closed loop system

$$a \cdot \ddot{\theta} + c \cdot \sin\theta = u = -\phi(\dot{\theta}) \cdot \Big[E(\theta, \dot{\theta}) - E_0\Big]$$

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Solutions $v^{\perp}(\cdot)$ approxim **How to find $\lambda(\cdot)$?** cs are defined by

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Example (Cont'd)

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Example (Cont'd)

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Example (Cont'd)

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$$\frac{d}{dt}[\lambda] \cdot \left(v_0^{\perp}\right)^T v_0^{\perp} + \lambda \cdot \left(v_0^{\perp}\right)^T \frac{d}{dt}\left[v_0^{\perp}\right] = \lambda \cdot \left(v_0^{\perp}\right)^T \mathcal{A}(t) v_0^{\perp}$$

Example (Cont'd)

The state space form $\dot{x} = f(x)$ of the closed loop system

$$a \cdot \ddot{\theta} + c \cdot \sin\theta = u = -\phi(\dot{\theta}) \cdot \left[E(\theta, \dot{\theta}) - E_0\right]$$

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$$\tfrac{d}{dt}\lambda = \boxed{\left[\ldots\right]} \cdot \lambda$$

Example (Cont'd)

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$$\frac{d}{dt}\left[\lambda \cdot v_0^{\perp}\right] = \frac{d}{dt}[\lambda] \cdot v_0^{\perp} + \lambda \cdot \frac{d}{dt}\left[v_0^{\perp}\right] = \mathcal{A}(t)\left[\lambda \cdot v_0^{\perp}\right]$$

$$\frac{d}{dt}\lambda = -\left[\dot{\theta}_0(t) \cdot \phi(\dot{\theta}_0(t))\right] \cdot \lambda$$

Example (Cont'd)

The state space form $\dot{x} = f(x)$ of the closed loop system

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$$\lambda(T) = \exp\left[-\int_0^T \dot{\theta}_0(t) \cdot \phi(\dot{\theta}_0(t))dt\right] \cdot \lambda(0)$$