# Lecture 6

## Motion Planning for Underactuated Systems: Case Studies

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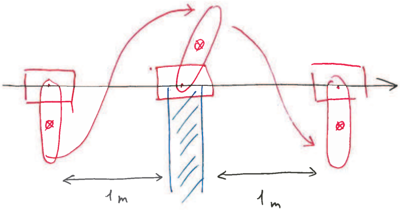
**Learning outcomes:** Problem formulation and settings for searching and planning behaviors of mechanical systems with one or two passive degrees of freedom. Examples

# Outline

1. Example: a pendulum on a cart
    - Moving a pendulum over an obstacle (a wall)
2. Example: a coin rolling on a table
    - Planning a rolling of a coin of along a circle

# Example: a pendulum on a cart

# Cart-pendulum system: moving a pendulum over a wall

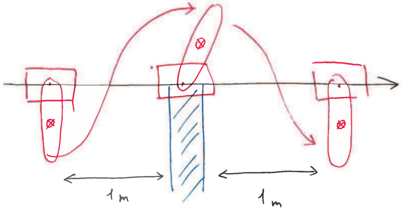


The task is to find a force applied to the cart such that in response the pendulum of the system comes over a wall without collision

One of conceptual solutions:

1. Push the cart to left to swing up the pendulum
2. Push the cart to right to pass the wall keeping the pendulum above the horizontal
3. Bring the cart to the final position simultaneously damping the pendulum oscillations

# Cart-pendulum system: moving a pendulum over a wall



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For the step 1 the dynamics with constant force $f_1 < 0$ are

$$\begin{aligned} 2\cdot\ddot{x} + \cos\theta\cdot\ddot{\theta} - \sin\theta\cdot\dot{\theta}^2 &= f_1 \\ \cos\theta\cdot\ddot{x} + \ddot{\theta} - g\cdot\sin\theta &= 0 \end{aligned}$$

$f_1$ should be sufficient to swing the pendulum above the horizontal

---

To find $f_1$, one needs to decouple the dynamics of $x$ and $\theta$

---

E.g. one can take $\ddot{x}$ from the 1st eqn and plug it into the 2nd eqn

$$\cos\theta\cdot\underbrace{\frac{1}{2}\left(f_1 - \cos\theta\cdot\ddot{\theta} + \sin\theta\cdot\dot{\theta}^2\right)}_{=\ddot{x}} + \ddot{\theta} - g\cdot\sin\theta = 0$$

---

Collecting the terms results in the dynamics of $\theta$-variable

$$\left(1 - \frac{1}{2}\cos^2\theta\right)\cdot\ddot{\theta} + \frac{1}{2}\cos\theta\cdot\sin\theta\cdot\dot{\theta}^2 + \left(\frac{1}{2}\cdot f_1\cdot\cos\theta - g\cdot\sin\theta\right) = 0$$

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# Cart-pendulum system: moving a pendulum over a wall

The initial conditions for a searched motion $\theta^*(\cdot)$ of the system

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are defined as

$$\theta^*(0)=\pi,\quad \dot{\theta}^*(0)=0$$

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How to find $f_1$ such that the solution $\theta^*(\cdot)$ will pass through

$$\theta^*(T)=a,\quad \dot{\theta}^*(T)=0 \text{ ???}$$

The $\theta$-dynamics has the integral of motion for any $f_1$ defined as

$$E_{red}(\theta(t),\dot{\theta}(t))=\tfrac{1}{2}\left(1-\tfrac{1}{2}\cos^2\theta(t)\right)\dot{\theta}^2(t)+g\cos\theta(t)+\tfrac{1}{2}f_1\sin\theta(t)\equiv C$$

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Hence

$$\begin{aligned} E_{red}(\theta^*(T), \dot{\theta}^*(T)) &= E_{red}(a, 0) = g\cdot\cos a + \frac{1}{2}\cdot f_1\cdot\sin a \\ &= E_{red}(\theta^*(0), \dot{\theta}^*(0)) = g\cdot\cos\pi + \frac{1}{2}\cdot f_1\cdot\sin\pi \end{aligned}$$

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Hence

$$g\cdot\cos a+\tfrac{1}{2}\cdot f_1\cdot\sin a=-g$$

$$f_1=-2\cdot\frac{1+\cos a}{\sin a}\cdot g$$

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**How to compute a time constant T?**

# Cart-pendulum system: moving a pendulum over a wall

To estimate the position and velocity of the cart at the end of this maneuver, one can observe that

$$\begin{aligned}
2\cdot\ddot{x}+\cos\theta\cdot\ddot{\theta}-\sin\theta\cdot\dot{\theta}^2 &= {\color{red}f_1}\\
&\Updownarrow\\
\tfrac{d}{dt}\left[2\cdot\dot{x}+\cos\theta\cdot\dot{\theta}\right] &= {\color{red}f_1}\\
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\left\{2\cdot\dot{x}(t)+\cos\theta(t)\cdot\dot{\theta}(t)\right\}-\left\{2\cdot\dot{x}(0)+\cos\theta(0)\cdot\dot{\theta}(0)\right\} &= {\color{red}f_1}\cdot t\\
&\Updownarrow\\
\left\{2\cdot x(t)+\sin\theta(t)\right\}-\left\{2\cdot x(0)+\sin\theta(0)\right\}-\\
\left\{2\cdot\dot{x}(0)+\cos\theta(0)\cdot\dot{\theta}(0)\right\}\cdot t &= \tfrac{1}{2}\cdot{\color{red}f_1}\cdot t^2
\end{aligned}$$

---

If the system was at rest in the beginning of the movement, then

$$2x(T)+\sin {\color{red}a}-2x(0)=\tfrac{1}{2}\cdot{\color{red}f_1}\cdot T^2,\qquad 2\dot{x}(T)={\color{red}f_1}\cdot T$$

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On the 2nd step, one needs to find a constant $f_2 > 0$ such that for the solution of the system

$$\begin{aligned} 2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 &= f_2 \\ \cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta &= 0 \end{aligned}$$

the position of the cart moves behind the wall while the pendulum stays above the horizontal for the same time interval.

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As before the behavior of the $\theta$-variable is defined by as a solution $\theta^*(\cdot)$ of the reduced dynamics

$$\left(1 - \tfrac{1}{2}\cos^2\theta\right) \cdot \ddot{\theta} + \tfrac{1}{2}\cos\theta \cdot \sin\theta \cdot \dot{\theta}^2 + \left(\tfrac{1}{2} \cdot f_2 \cdot \cos\theta - g \cdot \sin\theta\right) = 0$$

with initial conditions at

$$\theta^*(0) = a, \quad \dot{\theta}^*(0) = 0 \quad \text{with} \quad -\frac{\pi}{2} < a < \frac{\pi}{2}$$

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As for the 1st maneuver, one can use the integral of $\theta$-dynamics

$$E_{red}(\theta(t),\dot{\theta}(t)) = \frac{1}{2}\left(1-\frac{1}{2}\cos^2\theta(t)\right)\dot{\theta}^2(t)+g\cos\theta(t)+\frac{1}{2}f_2\sin\theta(t)\equiv C$$

for computing for the second maneuver

- time to reach the critical values of the angle: $\theta_c = \pm\frac{\pi}{2}$
- angular velocity of the pendulum at critical values of $\theta$

as functions of the input variable $f_2$.

---

**Homework:** find constants $f_1$ and $f_2$, which in combination create the requested behavior of the cart-pendulum.

---

One can investigate more complicated situations when

$$f_1 = f_{10}+f_{11}\cdot\theta+\cdots+f_{1k}\cdot\theta^k, \quad f_2 = f_{20}+f_{21}\cdot\theta+\cdots+f_{2m}\cdot\theta^m$$

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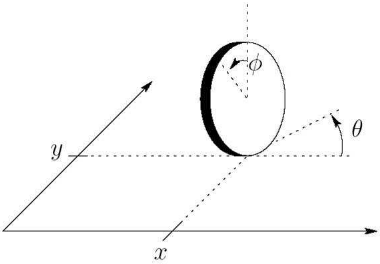
# A coin rolling on a table

# Example: planning a rolling of a coin



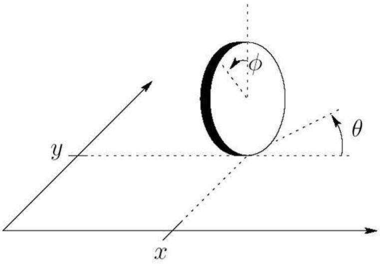
The equations of motion are

$$m\ddot{x} = F_x^c, \qquad m\ddot{y} = F_y^c, \qquad J\ddot{\theta} = u$$

Here $u$ is control; $F^c = [F_x^c;\ F_y^c]$ is the force due to the constraint

$$\dot{y}(t)\cos\theta(t) - \dot{x}(t)\sin\theta(t) \equiv 0, \quad \forall\, t$$

# Example: planning a rolling of a coin

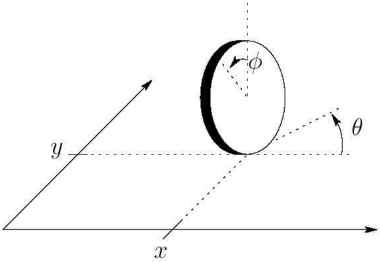


The equations of motion are

$$m\ddot{x} = \lambda \cdot \cos(\theta - \tfrac{\pi}{2}), \qquad m\ddot{y} = \lambda \cdot \sin(\theta - \tfrac{\pi}{2}), \qquad J\ddot{\theta} = u$$

Here $\lambda$ is amplitude of the constraint force.

# Example: planning a rolling of a coin

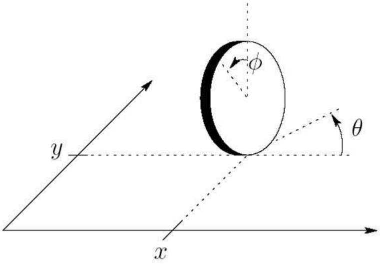


The equations of motion are

$$
\begin{aligned}
\ddot{x} &= -\left[\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta\right]\cdot\dot{\theta}\cdot\sin(\theta)\\
\ddot{y} &= \left[\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta\right]\cdot\dot{\theta}\cdot\cos(\theta)\\
J\ddot{\theta} &= u
\end{aligned}
$$

where constraint force is quadratic in velocities.

# Example: planning a rolling of a coin



We have already seen irreducible equations of motion of the form

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}\mathcal{L}\right] - \frac{\partial}{\partial q}\mathcal{L} = R(q,\dot{q}) + B(q)u, \qquad \boxed{R_i = \dot{q}^T r_i(q)\dot{q}}$$

Here $q \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $R(\cdot)$ is a vector of reaction forces.

# Example: planning a rolling of a coin of along a circle

Motion planning for the dynamical model

$$\begin{aligned}\ddot{x} &= -[\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta]\cdot\dot{\theta}\cdot\sin(\theta)\\ \ddot{y} &= [\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta]\cdot\dot{\theta}\cdot\cos(\theta)\\ J\ddot{\theta} &= u\end{aligned}$$

can be quite non-trivial.

# Example: planning a rolling of a coin of along a circle

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Find feasible motions of the system consistent with requirement

the center of mass should stay on a circle of radius $R$

# Example: planning a rolling of a coin of along a circle

Motion planning for the dynamical model

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can be quite non-trivial.

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I.e. along any such motion $[x_c(t), y_c(t), \theta_c(t)]$ the relations hold

$$\boxed{x_c(t) = R\cdot\cos\left(\theta_c(t) - \frac{\pi}{2}\right)} \qquad \boxed{y_c(t) = R\cdot\sin\left(\theta_c(t) - \frac{\pi}{2}\right)}$$

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Motion planning for the dynamical model

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I.e. along any such motion $[x_c(t), y_c(t), \theta_c(t)]$ the relations hold

$$x_c(t) = R\cdot\sin\theta_c(t) \qquad y_c(t) = -R\cdot\cos\theta_c(t)$$

# Example: planning a rolling of a coin of along a circle

Motion planning for the dynamical model

$$\begin{aligned} \ddot{x} &= -\left[\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta\right]\cdot\dot{\theta}\cdot\sin(\theta) \\ \ddot{y} &= \left[\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta\right]\cdot\dot{\theta}\cdot\cos(\theta) \\ J\ddot{\theta} &= u \end{aligned}$$

can be quite non-trivial.

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I.e. along any such motion $[x_c(t), y_c(t), \theta_c(t)]$ the relations hold

$$x_c(t) = R\cdot\sin\theta_c(t) \qquad y_c(t) = -R\cdot\cos\theta_c(t)$$

$$\dot{x}_c(t) = R\cdot\cos\theta_c(t)\cdot\dot{\theta}_c(t) \qquad \dot{y}_c(t) = R\cdot\sin\theta_c(t)\cdot\dot{\theta}_c(t)$$

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$$\boxed{\dot{x}_c(t) = R\cdot\cos\theta_c(t)\cdot\dot{\theta}_c(t)} \qquad \boxed{\dot{y}_c(t) = R\cdot\sin\theta_c(t)\cdot\dot{\theta}_c(t)}$$

$$\boxed{\ddot{x}_c = R\left[\cos\theta_c\ddot{\theta}_c - \sin\theta_c\dot{\theta}_c^2\right]} \qquad \boxed{\ddot{y}_c = R\left[\sin\theta_c\ddot{\theta}_c + \cos\theta_c\dot{\theta}_c^2\right]}$$

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Along a circular motion $[x_c(t), y_c(t), \theta_c(t)]$ of the system

$$\begin{aligned}
\ddot{x} &= -[\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta]\cdot\dot{\theta}\cdot\sin(\theta)\\
\ddot{y} &= [\dot{y}\cdot\sin\theta + \dot{x}\cdot\cos\theta]\cdot\dot{\theta}\cdot\cos(\theta)\\
J\ddot{\theta} &= u
\end{aligned}$$

the relations hold

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\ddot{x}_c &= R\left[\cos\theta_c\ddot{\theta}_c - \sin\theta_c\dot{\theta}_c^2\right]\\
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the relations hold

$$\begin{aligned} \boxed{\cos\theta_c\cdot\ddot{x}_c} &= \boxed{\cos\theta_c\cdot R\cdot\left[\cos\theta_c\ddot{\theta}_c - \sin\theta_c\dot{\theta}_c^2\right]} \\ \boxed{\sin\theta_c\cdot\ddot{y}_c} &= \boxed{\sin\theta_c\cdot R\cdot\left[\sin\theta_c\ddot{\theta}_c + \cos\theta_c\dot{\theta}_c^2\right]} \end{aligned}$$

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the relations hold

$$\begin{aligned} \boxed{\cos\theta_c\cdot\ddot{x}_c} &= \boxed{\cos\theta_c\cdot{\color{red}R}\cdot\left[\cos\theta_c\ddot{\theta}_c - \sin\theta_c\dot{\theta}_c^2\right]} \\ \boxed{\sin\theta_c\cdot\ddot{y}_c} &= \boxed{\sin\theta_c\cdot{\color{red}R}\cdot\left[\sin\theta_c\ddot{\theta}_c + \cos\theta_c\dot{\theta}_c^2\right]} \\ &\Downarrow \\ \cos\theta_c\cdot\ddot{x}_c + \sin\theta_c\cdot\ddot{y}_c &= {\color{red}R}\cdot\ddot{\theta}_c \\ &\Downarrow \\ 0 &= {\color{red}R}\cdot\ddot{\theta}_c \end{aligned}$$

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Any circular motion $[x_c(t), y_c(t), \theta_c(t)]$ of the system

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has the form

$$\begin{aligned} \theta_c(t) &= \omega_c \cdot t + \theta_0 \\ x_c(t) &= R \cdot \sin\theta_c(t) \\ y_c(t) &= -R \cdot \cos\theta_c(t) \\ u_c(t) &= 0 \end{aligned}$$

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**How to find new coordinates for the dynamics that are zero on the nominal motion and nonzero away from it? Regulating them to zero helps stabilizing such motion orbitally**

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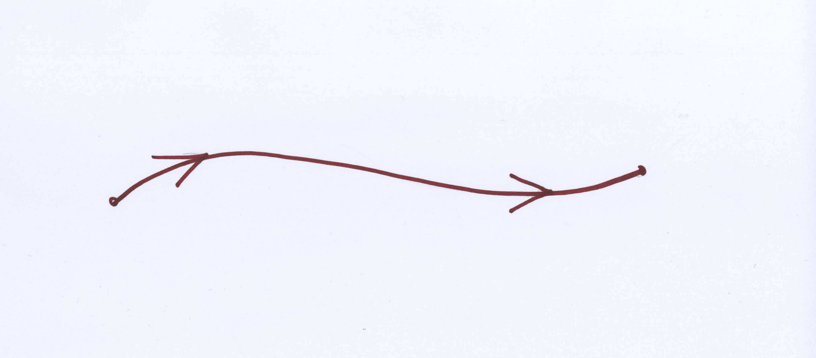
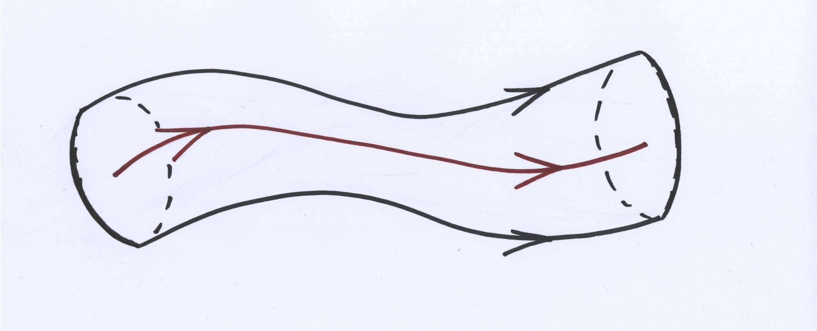
**How to find new coordinates for the dynamics that are zero on the nominal motion and nonzero away from it? Regulating them to zero helps stabilizing such motion orbitally**

# Geometrical interpretation



Given a trajectory of a nominal motion

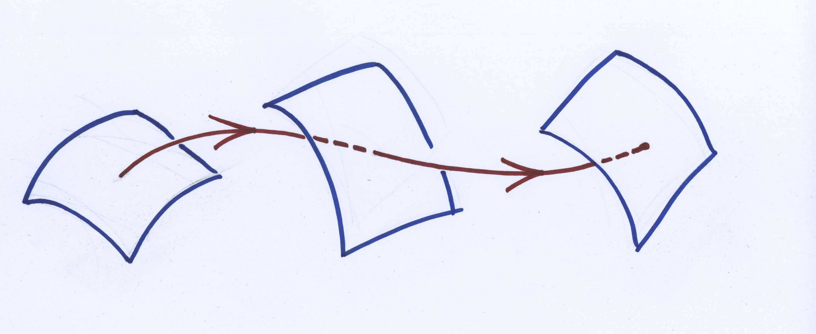
# Geometrical interpretation



We would like to analyze properties of the dynamics

in its tubing vicinity

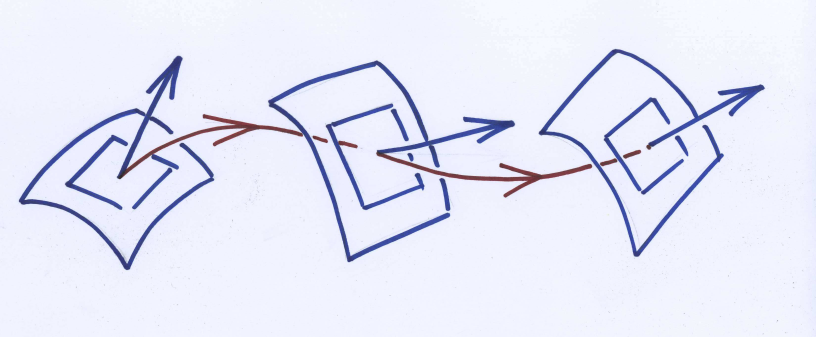
# Geometrical interpretation



Introduce a family of dis-joint transverse surfaces

that are continuously slicing this vicinity

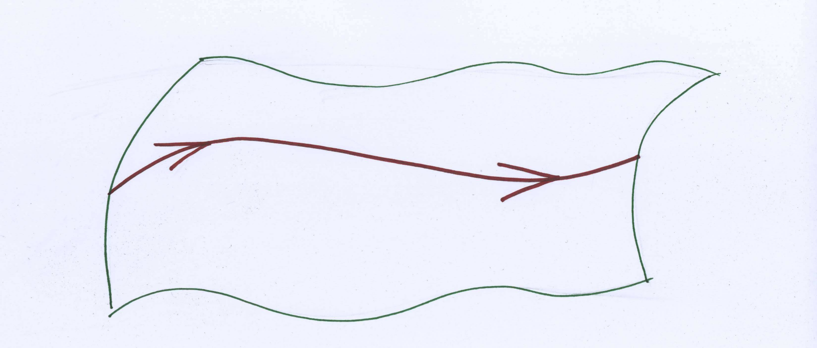
# Geometrical interpretation



For the linearization of the dynamics the surfaces

are substituted by tangent planes

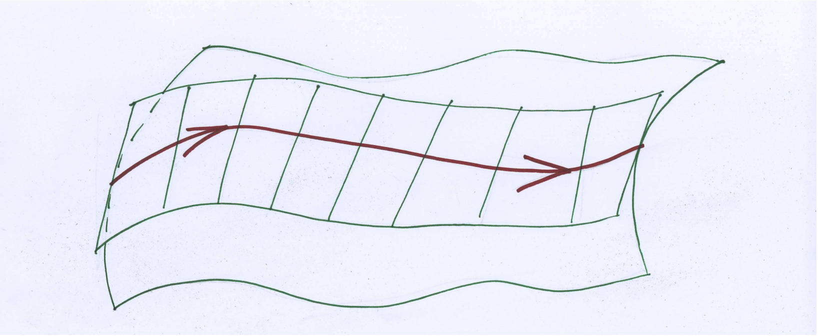
# Geometrical interpretation



If the dynamics have some invariants,

then they define a manifold

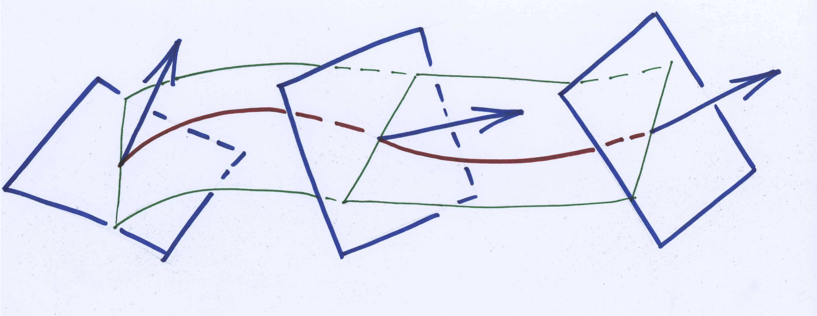
# Geometrical interpretation



For the linearization we consider the linear subspaces

that are tangent to to the trajectory along this manifold

# Geometrical interpretation



Evolution of coordinates on these linear subspaces will define

linearization of transverse coordinates with nontrivial behavior