

Lecture 4

Introduction to Stability of Mechanical Systems

Anton Shiriaev

Norwegian University of Science and Technology

Trondheim, Norway

Learning outcomes: Tools for analysis of stability of a motion of a nonlinear mechanical system. Examples.

1. Example: a mathematical pendulum
2. Example: a pendulum on a cart
3. Lagrange-Dirichlet theorem
4. Example: restricted 3 body problem

Example: a mathematical pendulum

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Let us investigate a stability of equilibriums of the system

$$\ddot{\theta} + \frac{g}{l} \cdot \sin \theta = 0. \quad (1)$$

There are two equilibriums (mod 2π): $\theta_e = 0$ and $\theta_e = \pi$.

Linearization of the dynamics at $\theta_e = \pi$ results in

$$\ddot{z} - \frac{g}{l} \cdot z = 0.$$

The equilibrium of this linear system is unstable. Therefore, the equilibrium $\theta_e = \pi$ of the nonlinear system (1) is unstable as well.

Linearization of the dynamics at $\theta_e = 0$ results in

$$\ddot{z} + \frac{g}{l} \cdot z = 0.$$

This linear system has the center at the origin and the nonlinear dynamics (1) has the first integral. Therefore, the nonlinear system (1) has the center at $\theta_e = 0$ and this equilibrium is stable!

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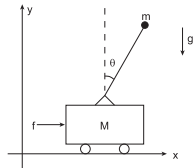
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Example: a pendulum on a cart

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Consider a pendulum (a point of a mass m at the distance l from the suspension point) attached to a cart of a mass M , which freely moves on the horizontal with $f = 0$.



When $M = m = l = 1$ the dynamics in coordinates (x, θ) are

$$2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 = 0 \quad (= f)$$

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

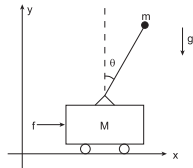
and have two sets of equilibriums:

- the pendulum is up, the cart is in any position: $\theta_e = 0, \forall x_e$
- the pendulum is down, the cart is in any position: $\theta_e = \pi, \forall x_e$

Let us investigate their stability!

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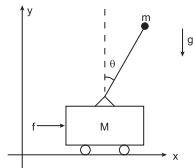
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The system $2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 = 0$

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

has two integrals of motion (CoM): the total energy

$$E = \frac{1}{2} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}^T \begin{bmatrix} 2 & \cos \theta \\ \cos \theta & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + g \cdot \cos \theta$$

and the momentum conjugated to the x -coordinate

$$P(x(t), \theta(t), \dot{x}(t), \dot{\theta}(t)) = 2 \cdot \dot{x}(t) + \cos \theta(t) \cdot \dot{\theta}(t)$$

The last relation can be integrated again and leads to

$$\begin{aligned} x(t) &= x(0) + \frac{1}{2} \sin \theta(0) - \frac{1}{2} \sin \theta(t) + \frac{1}{2} P(x(0), \theta(0), \dot{x}(0), \dot{\theta}(0)) \cdot t \\ &= -\frac{1}{2} \sin \theta(t) + C_0 + C_1 \cdot t \end{aligned}$$

If one is able to compute $\theta(\cdot)$, then the formula gives $x(\cdot)$!

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For decoupling dynamics of the θ -variable in the system

$$2 \cdot \ddot{x} + \cos \theta \cdot \ddot{\theta} - \sin \theta \cdot \dot{\theta}^2 = 0$$

$$\cos \theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin \theta = 0$$

one can use the second equation.

Indeed

$$\cos \theta \cdot \overbrace{\frac{1}{2} \left(\sin \theta \cdot \dot{\theta}^2 - \cos \theta \cdot \ddot{\theta} \right)}^{= \ddot{x}} + \ddot{\theta} - g \cdot \sin \theta = 0$$

substituting and collecting the terms one derives the equation

$$\left(1 - \frac{1}{2} \cos^2 \theta\right) \cdot \ddot{\theta} + \frac{1}{2} \cos \theta \cdot \sin \theta \cdot \dot{\theta}^2 - g \cdot \sin \theta = 0$$

What can one say on the dynamics of this system
in a vicinity of its equilibriums at 0 and at π ?

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$$\ddot{z} + \left[\frac{d}{d\theta} \left(\frac{-g \cdot \sin \theta}{1 - \frac{1}{2} \cos^2 \theta} \right) \right] \bigg|_{\theta=0} \cdot z = \ddot{z} - 2 \cdot g \cdot z = 0$$

Its linearization at $\theta_e = \pi$ is stable

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In addition the system has the first integral

$$E_{red} = \frac{1}{2} \left(1 - \frac{1}{2} \cos^2 \theta\right) \cdot \dot{\theta}^2 + g \cdot \cos \theta$$

The nonlinear system has the center at $\theta_e = \pi$!

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To conclude:

- Any cart pendulum solution can be written as

$$x(t) = \frac{1}{2} \sin \theta(t) + C_0 + C_1 t$$

with $\theta(t)$ being a solution of reduced dynamics.

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- In a vicinity of the upright equilibrium $[\theta_e = 0, x = x_e]$ the reduced dynamics is hyperbolic, therefore **any of upright equilibriums is unstable**.
- In a vicinity of the downward equilibrium $[\theta_e = \pi, x = x_e]$ the reduced dynamics is stable, but $x(t)$ will drift with $C_1 \neq 0$. Hence **any of downward equilibriums is unstable as well**.

Lagrange-Dirichlet theorem

Theorem (1788)

If at the position of an isolated equilibrium of a conservative mechanical system with holonomic constraints the potential energy Π has a strict minimum, then this equilibrium is stable.

Example: restricted 3 body problem

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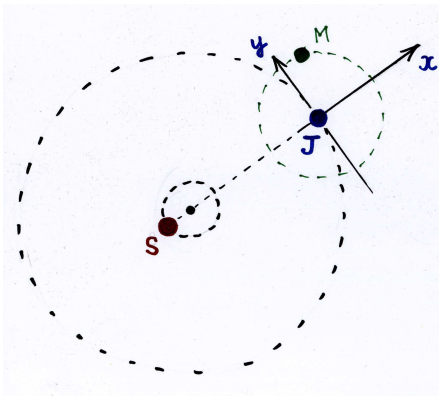
Equations of motion for the position of the Moon in rotating coordinate frame are

$$\begin{cases} \ddot{x} - 2m\dot{y} = \frac{\partial}{\partial x} F \\ \ddot{y} + 2m\dot{x} = \frac{\partial}{\partial y} F \end{cases}$$

Here

$$F = \frac{\kappa}{\sqrt{x^2 + y^2}} + \frac{3}{2}m^2x^2$$

m, κ are positive constants.



The system has the invariant:

$$I = \dot{x}^2 + \dot{y}^2 - 2F(x, y) + C$$

Task: Analyze the dynamics in a vicinity of the periodic motion

Example: restricted 3 body problem

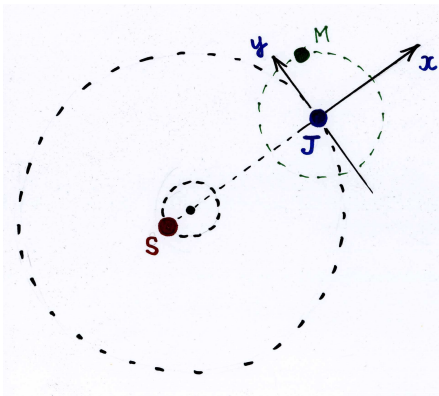
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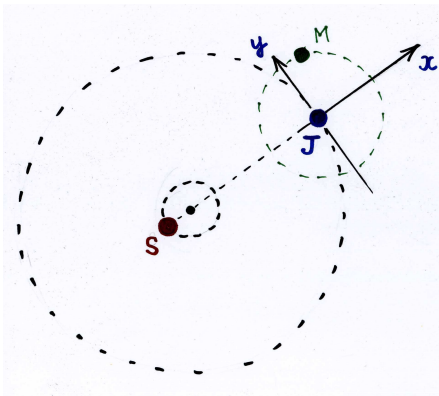
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Task: Analyze the dynamics in a vicinity of the periodic motion

Example: elements of theory of G.W. Hill

Denote as $[x_p(t), y_p(t)]$ the periodic solution

Perturbed solutions $[x_p(t) + \delta x(t), y_p(t) + \delta y(t)]$ defined by

$$\begin{aligned}\frac{d^2}{dt^2} [\delta x] - 2m \frac{d}{dt} [\delta y] &= \\ &= \left[\frac{\partial^2}{\partial x^2} F(x_p(t), y_p(t)) \right] \delta x + \left[\frac{\partial^2}{\partial x \partial y} F(x_p(t), y_p(t)) \right] \delta y\end{aligned}$$

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The integral Jacobi $I(\cdot)$ gives another relation

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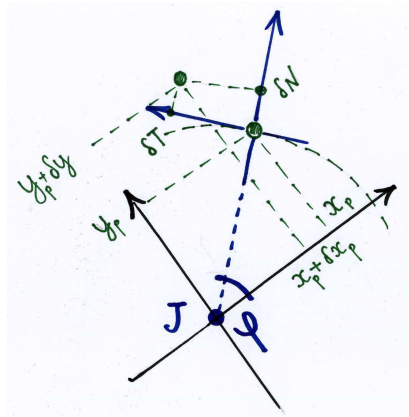
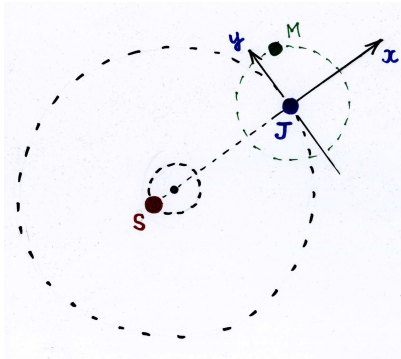
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Example: elements of theory of G.W. Hill



Transform of coordinates into normal (δN) and tangent (δT)

$$\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \delta T \\ \delta N \end{bmatrix}$$

Example: elements of theory of G.W. Hill

In a vicinity of the motion the original coordinates

$$\left[x, y, \dot{x}, \dot{y} \right]$$

are changed into

$$\left[\phi, I, N, \dot{N} \right]$$

The linearization of $\phi(\cdot)$ is not important: it perpetually rotates

The linearization of $I(\cdot)$ is straightforward: $\frac{d}{dt} [\delta I] \equiv 0$

The linearization of $[N, \dot{N}]$ is the famous Hill's equation

$$\frac{d^2}{dt^2} [\delta N] + \Phi(t) \delta N = 0$$

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The linearization of $I(\cdot)$ is straightforward: $\frac{d}{dt} [\delta I] \equiv 0$

The linearization of $[N, \dot{N}]$ is the famous Hill's equation

$$\frac{d^2}{dt^2} [\delta N] + \Phi(t) \delta N = 0$$

Example: elements of theory of G.W. Hill

In a vicinity of the motion the original coordinates

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are changed into

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Outcome of the example

Analysis of dynamics in a vicinity of the motion's orbit requires:

- Decomposition of coordinates into
 - **transverse** to the trajectory ($\dim = 2n - 1$)
 - **along** the trajectory ($\dim = 1$)

In the example they are

$$\begin{bmatrix} I, N, \dot{N} \end{bmatrix} \quad \text{and} \quad \phi$$

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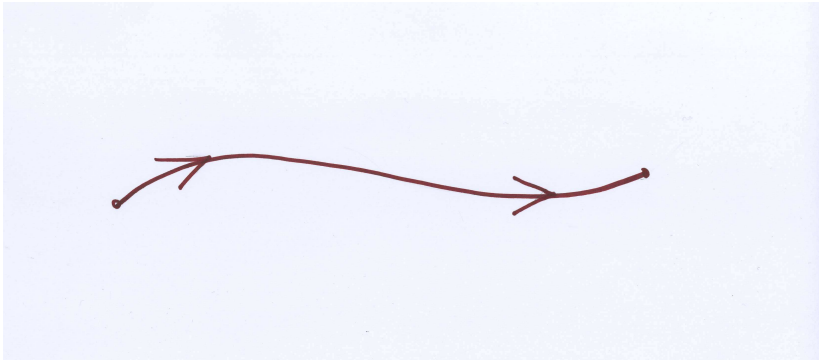
In the example they are

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-
- Presence of invariants allows to reduce a number of transverse coordinates with non-trivial dynamics.

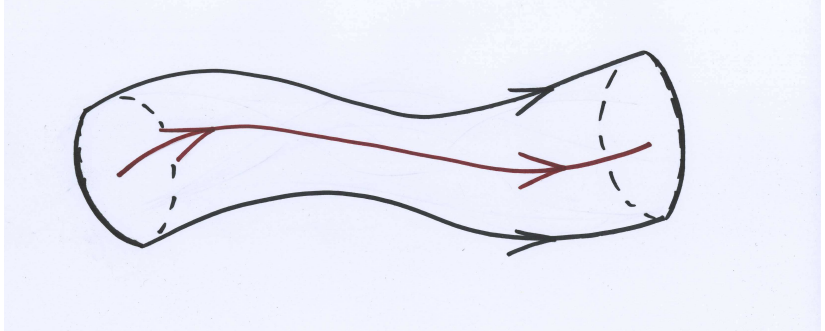
In the example the integral Jacobi $I(\cdot)$ is excluded.

Outcome of the example: geometrical interpretation



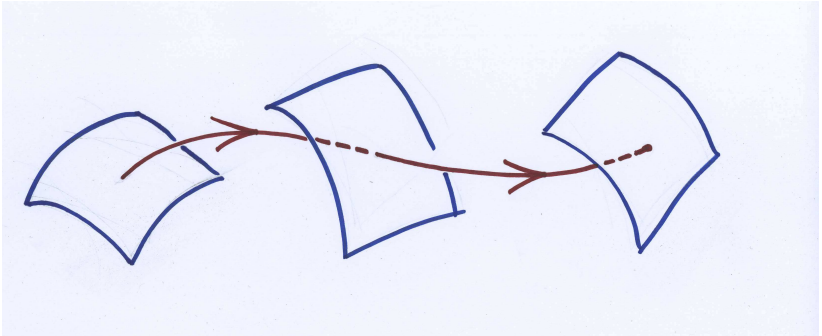
Given a trajectory of a nominal motion

Outcome of the example: geometrical interpretation



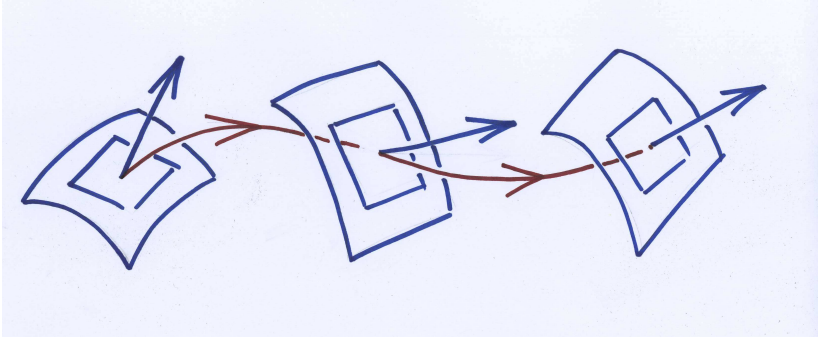
We would like to analyze properties of the dynamics
in its tubing vicinity

Outcome of the example: geometrical interpretation



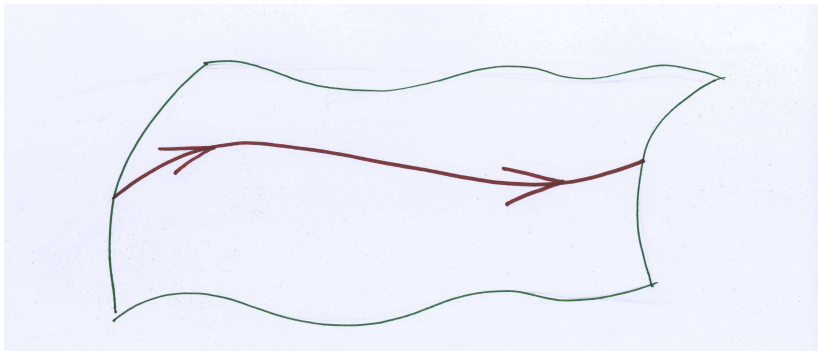
Introduce a family of dis-joint transverse surfaces
that are continuously slicing this vicinity

Outcome of the example: geometrical interpretation



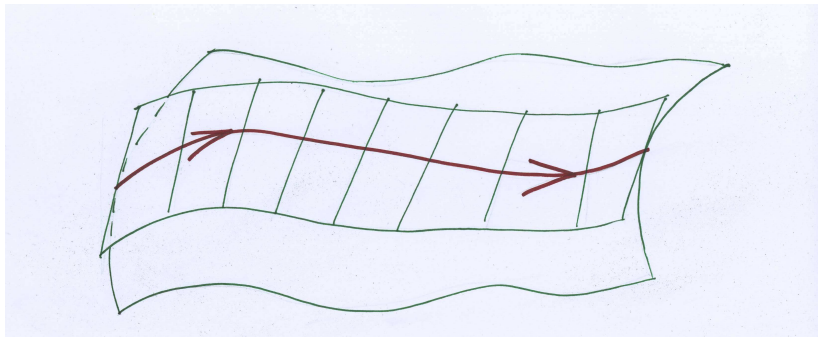
For the linearization of the dynamics the surfaces
are substituted by tangent planes

Outcome of the example: geometrical interpretation



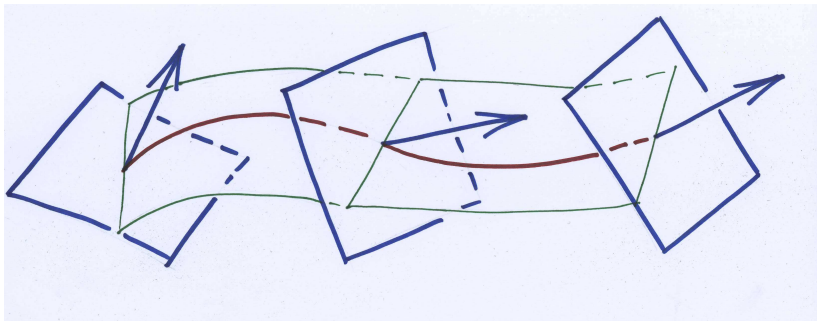
If the dynamics have some invariants,
then they define a manifold

Outcome of the example: geometrical interpretation



For the linearization we consider the linear subspaces
that are tangent to the trajectory along this manifold

Outcome of the example: geometrical interpretation



Evolution of coordinates on these linear subspaces will define linearization of transverse coordinates with nontrivial behavior