

# Lecture 8

## Orbital Stability and Stabilization for Underactuated Systems

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**Learning outcomes:** Transverse coordinates and transverse linearization for forced motions of underactuated mechanical system. Examples

1. Transverse coordinates for mechanical systems

2. Transverse linearization for mechanical systems

## **Transverse coordinates**

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## Alternative Representations of a Motion

Given a motion  $q^*(t)$ ,  $t \in [0, T]$  of a mechanical underactuated system

$$\frac{d}{dt} \left[ \frac{\partial \mathcal{L}}{\partial \dot{q}} \right] - \frac{\partial \mathcal{L}}{\partial q} = B(q)u, \quad q \in \mathbb{R}^n, \quad u \in \mathbb{R}^{n-1}$$

obtained in response to the input signal  $u^*(t)$ ,  $t \in [0, T]$

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if the motion admits kinematic relations

$$q_1 = \phi_1(\theta), \quad q_2 = \phi_2(\theta), \quad \dots, \quad q_n = \phi_n(\theta)$$

valid between the coordinates  $q_i$  and a new scalar variable  $\theta$

---

then there is a scalar system with coefficients defined by  $\mathcal{L}$  and  $\{\phi_i\}_{i=1}^n$

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0 \quad \text{and its solution } \theta^*(\cdot)$$

---

which allows re-writing the motion  $q^*(\cdot)$  on the time interval  $[0, T]$  as

$$q_1^*(t) = \phi_1(\theta^*(t)), \quad q_2^*(t) = \phi_2(\theta^*(t)), \quad \dots, \quad q_n^*(t) = \phi_n(\theta^*(t))$$

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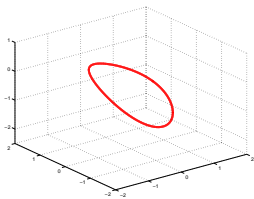
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## Transverse Coordinates for a Motion :



Given a  $T$ -periodic motion of  $n$ -DOF MS

$$q^*(t) = [q_1^*(t); q_2^*(t); \dots; q_n^*(t)] = q^*(t+T),$$

a scalar  $\theta$ , the functions for its representation

$$\phi_1(\cdot), \quad \phi_2(\cdot), \quad \dots, \quad \phi_n(\cdot)$$

The  $(n+1)$  coordinates for describing the dynamics in a vicinity of  $q^*(t)$

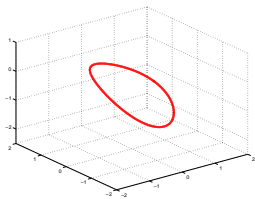
$$\theta, \quad y_1 := q_1 - \phi_1(\theta), \quad \dots, \quad y_n := q_n - \phi_n(\theta)$$

are excessive! And, therefore, they are not independent and one variable is redundant!

Then one of candidates for transverse coordinates  $x_{\perp}$  for a motion of the system with  $x = (q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)^T$  is given by  $(2n-1)$ -quantities

$$x_{\perp} = [I(\theta, \dot{\theta}, \theta^*(0), \dot{\theta}^*(0)); y_1; \dots; y_{n-1}; \dot{y}_1, \dots; \dot{y}_{n-1}]$$

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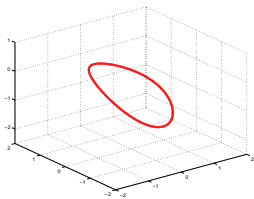
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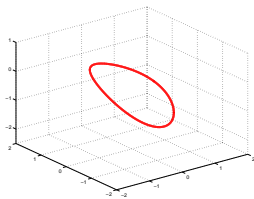
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## **Transverse linearization**

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## Steps in Computing Transverse Linearization

Given a  $T$ -periodic motion  $q^*(t) = (q_1^*(t), q_2^*(t), \dots, q_n^*(t))^T$  of the mechanical system

$$\frac{d}{dt} \left[ \frac{\partial \mathcal{L}}{\partial \dot{q}} \right] - \frac{\partial \mathcal{L}}{\partial q} = B(q)u$$

Step 1: Rewrite the dynamics in the coordinates:  $\theta, y$  with

$$y_1 = q_1 - \phi_1(\theta), \quad \dots, \quad y_{n-1} = q_{n-1} - \phi_{n-1}(\theta)$$

Step 2: Define a feedback transformation

$$u = (u_1, \dots, u_{n-1}) \rightarrow v = (v_1, \dots, v_{n-1})$$

such that the dynamics of  $y$  become linear

$$\ddot{y} = v, \quad \ddot{\theta} = N(\theta, \dot{\theta}, y, \dot{y}, \ddot{y})$$

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## Steps in Computing Transverse Linearization: $\theta$ -dynamics

Since the  $\theta$ -dynamics with constraints  $y = \dot{y} = \ddot{y} = \mathbf{v} = 0$

$$\ddot{\theta} = N(\theta, \dot{\theta}, y, \dot{y}, \ddot{y})$$

become

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

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Therefore, the dynamics of  $\theta$ -variable can be rewritten as

$$\begin{aligned} \alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) &= g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})y + \\ &\quad + g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\dot{y} + g_v(\theta, \dot{\theta}, y, \dot{y})\ddot{y} \end{aligned}$$

The functions  $g_y(\cdot)$ ,  $g_{\dot{y}}(\cdot)$  and  $g_v(\cdot)$  are to be found based on

- the Lagrangian  $\mathcal{L}(\cdot)$  of the system;
- the functions  $\phi_1(\cdot), \dots, \phi_{n-1}(\cdot)$  representing the motion

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## Computing Transverse Linearization: Passivity Relation for $I(\cdot)$

The mechanical system written in  $y, \theta$ -coordinates is

$$\begin{aligned}\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) &= g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})y + \\ &+ g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\dot{y} + g_v(\theta, \dot{\theta}, y, \dot{y})v \\ \ddot{y} &= v\end{aligned}$$

We are searching for linearization of transverse dynamics

$$x_{\perp} = \left[ I(\theta, \dot{\theta}, \theta^*(0), \dot{\theta}^*(0)); y; \dot{y} \right]$$

around the target solution:  $\theta(t) = \theta^*(t)$ ,  $y(t) \equiv 0$

To do so, one can use the passivity relation for  $\frac{d}{dt}I(\cdot)$  found previously

$$\begin{aligned}\frac{d}{dt}I(\theta, \dot{\theta}, a, b) &\equiv \dot{\theta} \left\{ \frac{2}{\alpha(\theta)} W - \frac{2\beta(\theta)}{\alpha(\theta)} I(\theta, \dot{\theta}, a, b) \right\} \\ &\equiv \dot{\theta} \left\{ \frac{2}{\alpha(\theta)} \left[ g_y(\cdot)y + g_{\dot{y}}(\cdot)\dot{y} + g_v(\cdot)v \right] - \frac{2\beta(\theta)}{\alpha(\theta)} I(\theta, \dot{\theta}, a, b) \right\}\end{aligned}$$

## Computing Transverse Linearization: Passivity Relation for $I(\cdot)$

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## Transverse Linearization: the Final Form

The linear control system with  $(2n - 1)$  states and  $(n - 1)$  control inputs

$$\begin{aligned} \frac{d}{dt} l_{\heartsuit}(t) &= \frac{2\dot{\theta}^*(t)}{\alpha(\theta^*(t))} \left\{ \left[ \tilde{g}_y(t)y_{\heartsuit}(t) + \tilde{g}_{\dot{y}}(t)\dot{y}_{\heartsuit}(t) + \tilde{g}_v(t)v_{\heartsuit}(t) \right] - \right. \\ &\quad \left. - \beta(\theta^*(t))l_{\heartsuit}(t) \right\} \\ \ddot{y}_{\heartsuit}(t) &= v_{\heartsuit}(t) \end{aligned}$$

## Transverse Linearization: the Final Form

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with the functions

$$\begin{aligned} \tilde{g}_y(t) &= g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y}) \Big|_{\substack{\theta = \theta^*(t), \dot{\theta} = \dot{\theta}^*(t) \\ \ddot{\theta} = \ddot{\theta}^*(t), y = \dot{y} = 0}} \\ \tilde{g}_{\dot{y}}(t) &= g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y}) \Big|_{\substack{\theta = \theta^*(t), \dot{\theta} = \dot{\theta}^*(t) \\ \ddot{\theta} = \ddot{\theta}^*(t), y = \dot{y} = 0}} \\ \tilde{g}_v(t) &= g_v(\theta, \dot{\theta}, y, \dot{y}) \Big|_{\substack{\theta = \theta^*(t), \dot{\theta} = \dot{\theta}^*(t) \\ y = \dot{y} = 0}} \end{aligned}$$

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### Transverse Linearization of the Mechanical System in a Vicinity of the Motion $q^*(\cdot)$

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Any **linear** feedback controller of the form

$$v_{\heartsuit}(t) = K(t) \begin{bmatrix} l_{\heartsuit} \\ y_{\heartsuit} \\ \dot{y}_{\heartsuit} \end{bmatrix}, \quad K(t) = K(t + T),$$

that stabilizes the origin of the linear control system can be transformed into a **nonlinear controller that stabilizes orbitally** the nominal periodic motion  $q^*(\cdot)$  of the nonlinear underactuated mechanical system