Lecture 8

Transverse Linearization and Stabilization of Transverse Dynamics for Forced Motions of Underactuated Mechanical Systems

Anton Shiriaev
Norwegian University of Science and Technology
Trondheim, Norway

Learning outcomes: Families of motion dependent transverse coordinates and procedures for computing transverse linearization for forced motions of underactuated mechanical system. Examples

Outline

1. Transverse coordinates for a forced motion of a mechanical system (Part I)

2. Transverse linearization for a forced motion of a mechanical system (Part I)

3. Example: a cart-pendulum system
   - Transverse coordinates for a forced oscillation around upward equilibrium of the pendulum
   - Linearization of transverse coordinates for a forced motion of the system

Transverse coordinates (Part I)

On Alternative (Nested) Representations of a Motion

Given a motion $q^*(t)$, $t \in [0, T]$ of an underactuated mechanical system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = R_0(q) + R_2(q, \dot{q}) + B(q)u, \qquad q \in \mathbb{R}^n, \; u \in \mathbb{R}^{n-1}$$

obtained in response to the input signal $u^*(t)$, $t \in [0, T]$

If the motion admits kinematic relations

$$q_1 = \phi_1(\theta), \qquad q_2 = \phi_2(\theta), \qquad \ldots, \qquad q_n = \phi_n(\theta)$$

valid between the coordinates q_i and a new scalar variable θ on the nominal forced behavior

---

Then there is a scalar system with coefficients defined by $\mathcal{L}(\cdot)$, $R_0(\cdot)$, $R_2(\cdot)$, $B(\cdot)$ and $\{\phi_i(\cdot)\}_{i=1}^n$

$$\alpha(\theta) \cdot \ddot{\theta} + \beta(\theta) \cdot \dot{\theta}^2 + \gamma(\theta) = 0 \qquad \text{and its solution} \qquad \theta^*(\cdot)$$

---

which allows re-writing the motion $q^*(\cdot)$ on the time interval $[0, T]$ as

$$q_1^*(t) = \phi_1(\theta^*(t)), \qquad q_2^*(t) = \phi_2(\theta^*(t)), \qquad \ldots, \qquad q_n^*(t) = \phi_n(\theta^*(t))$$

On Alternative (Nested) Representations of a Motion

Given a motion $q^*(t)$, $t \in [0, T]$ of an underactuated mechanical system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = R_0(q) + R_2(q, \dot{q}) + B(q)u, \qquad q \in \mathbb{R}^n, \; u \in \mathbb{R}^{n-1}$$

obtained in response to the input signal $u^*(t)$, $t \in [0, T]$

If the motion admits kinematic relations

$$q_1 = \phi_1(\theta), \qquad q_2 = \phi_2(\theta), \qquad \ldots, \qquad q_n = \phi_n(\theta)$$

valid between the coordinates q_i and a new scalar variable θ on the nominal forced behavior

---

Then there is a scalar system with coefficients defined by $\mathcal{L}(\cdot)$, $R_0(\cdot)$, $R_2(\cdot)$, $B(\cdot)$ and $\{\phi_i(\cdot)\}_{i=1}^n$

$$\alpha(\theta) \cdot \ddot{\theta} + \beta(\theta) \cdot \dot{\theta}^2 + \gamma(\theta) = 0 \qquad \text{and its solution} \qquad \theta^*(\cdot)$$

---

which allows re-writing the motion $q^*(\cdot)$ on the time interval $[0, T]$ as

$$q_1^*(t) = \phi_1(\theta^*(t)), \qquad q_2^*(t) = \phi_2(\theta^*(t)), \qquad \ldots, \qquad q_n^*(t) = \phi_n(\theta^*(t))$$

On Alternative (Nested) Representations of a Motion

Given a motion $q^*(t)$, $t \in [0, T]$ of an underactuated mechanical system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = R_0(q) + R_2(q, \dot{q}) + B(q)u, \qquad q \in \mathbb{R}^n, \ u \in \mathbb{R}^{n-1}$$

obtained in response to the input signal $u^*(t)$, $t \in [0, T]$

If the motion admits kinematic relations

$$q_1 = \phi_1(\theta), \qquad q_2 = \phi_2(\theta), \qquad \ldots, \qquad q_n = \phi_n(\theta)$$

valid between the coordinates q_i and a new scalar variable θ on the nominal forced behavior

---

Then there is a scalar system with coefficients defined by $\mathcal{L}(\cdot)$, $R_0(\cdot)$, $R_2(\cdot)$, $B(\cdot)$ and $\{\phi_i(\cdot)\}_{i=1}^n$

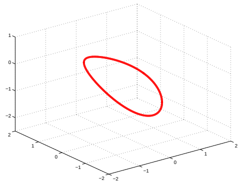
$$\alpha(\theta) \cdot \ddot{\theta} + \beta(\theta) \cdot \dot{\theta}^2 + \gamma(\theta) = 0 \qquad \text{and its solution} \qquad \theta^*(\cdot)$$

---

which allows re-writing the motion $q^*(\cdot)$ on the time interval $[0, T]$ as

$$q_1^*(t) = \phi_1(\theta^*(t)), \qquad q_2^*(t) = \phi_2(\theta^*(t)), \qquad \ldots, \qquad q_n^*(t) = \phi_n(\theta^*(t))$$

Transverse Coordinates for a Motion :



Given a T-periodic motion of n-dof mechanical system

$$q^\star(t) = \Big[q_1^\star(t);\ q_2^\star(t);\ \ldots;\ q_n^\star(t)\Big] = q^\star(t+T),$$

a scalar θ, and the n-smooth functions for its nested representation

$$\phi_1(\cdot), \quad \phi_2(\cdot), \quad \ldots, \quad \phi_n(\cdot)$$

The $(n+1)$ coordinates for describing the dynamics in a vicinity of $q^\star(t)$

$$\theta, \quad y_1 := q_1 - \phi_1(\theta), \quad \ldots, \quad y_n := q_n - \phi_n(\theta)$$

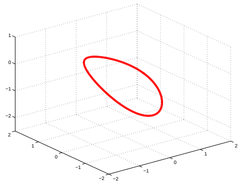
are excessive! And, therefore, they are not independent and atleast one variable is redundant!

Suppose the variables θ, $y = [y_1; \ldots; y_{n-1}]$ are independent

Then one of candidates for transverse coordinates $x_\perp$ for a motion of the system is given by

$$x_\perp = \Big[I(\theta, \dot\theta, \theta^\star(0), \dot\theta^\star(0));\ y_1;\ \ldots;\ y_{n-1};\ \dot y_1;\ \ldots;\ \dot y_{n-1}\Big]$$

Transverse Coordinates for a Motion :



Given a T-periodic motion of n-dof mechanical system

$$q^\star(t) = \left[q_1^\star(t);\ q_2^\star(t);\ \ldots;\ q_n^\star(t)\right] = q^\star(t+T),$$

a scalar θ, and the n-smooth functions for its nested representation

$$\phi_1(\cdot), \quad \phi_2(\cdot), \quad \ldots, \quad \phi_n(\cdot)$$

The $(n+1)$ coordinates for describing the dynamics in a vicinity of $q^\star(t)$

$$\theta, \quad y_1 := q_1 - \phi_1(\theta), \quad \ldots, \quad y_n := q_n - \phi_n(\theta)$$

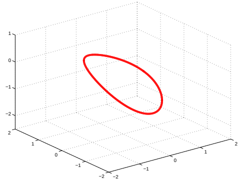
are excessive! And, therefore, they are not independent and atleast one variable is redundant!

Suppose the variables θ, $y = [y_1;\ \ldots;\ y_{n-1}]$ are independent

---

Then one of candidates for transverse coordinates $x_\perp$ for a motion of the system is given by

$$x_\perp = \left[I(\theta, \dot\theta, \theta^\star(0), \dot\theta^\star(0));\ y_1;\ \ldots;\ y_{n-1};\ \dot y_1;\ \ldots;\ \dot y_{n-1}\right]$$



Given a T-periodic motion of n-dof mechanical system

$$q^\star(t) = \left[q_1^\star(t);\ q_2^\star(t);\ \ldots;\ q_n^\star(t)\right] = q^\star(t+T),$$

a scalar θ, and the n-smooth functions for its nested representation

$$\phi_1(\cdot), \quad \phi_2(\cdot), \quad \ldots, \quad \phi_n(\cdot)$$

The $(n+1)$ coordinates for describing the dynamics in a vicinity of $q^\star(t)$

$$\theta, \quad y_1 := q_1 - \phi_1(\theta), \quad \ldots, \quad y_n := q_n - \phi_n(\theta)$$

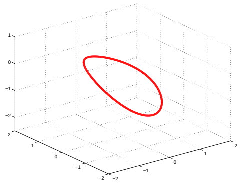
are excessive! And, therefore, they are not independent and atleast one variable is redundant!

Suppose the variables θ, $y = [y_1; \ldots; y_{n-1}]$ are independent

Then one of candidates for transverse coordinates $x_\perp$ for a motion of the system is given by

$$x_\perp = \left[I(\theta, \dot\theta, \theta^\star(0), \dot\theta^\star(0));\ y_1;\ \ldots;\ y_{n-1};\ \dot y_1;\ \ldots;\ \dot y_{n-1}\right]$$

Transverse Coordinates for a Motion :



Given a T-periodic motion of n-dof mechanical system

$$q^\star(t) = \left[q_1^\star(t);\ q_2^\star(t);\ \ldots;\ q_n^\star(t)\right] = q^\star(t+T),$$

a scalar θ, and the n-smooth functions for its nested representation

$$\phi_1(\cdot), \quad \phi_2(\cdot), \quad \ldots, \quad \phi_n(\cdot)$$

The $(n+1)$ coordinates for describing the dynamics in a vicinity of $q^\star(t)$

$$\theta, \quad y_1 := q_1 - \phi_1(\theta), \quad \ldots, \quad y_n := q_n - \phi_n(\theta)$$

are excessive! And, therefore, they are not independent and atleast one variable is redundant!

Suppose the variables θ, $y = [y_1;\ \ldots;\ y_{n-1}]$ are independent

---

Then one of candidates for transverse coordinates $x_\perp$ for a motion of the system is given by

$$x_\perp = \left[I(\theta, \dot\theta, \theta^\star(0), \dot\theta^\star(0));\ y_1;\ \ldots;\ y_{n-1};\ \dot y_1,\ \ldots;\ \dot y_{n-1}\right]$$

Transverse linearization (Part I)

Steps in Computing Transverse Linearization

Given a T-periodic motion $q^\star(t) = \left(q_1^\star(t),\, q_2^\star(t),\, \ldots,\, q_n^\star(t)\right)^T$ of the mechanical system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = R_0(q) + R_2(q, \dot{q}) + B(q)u$$

Step 1: Rewrite the dynamics in the coordinates: θ, y with

$$y_1 = q_1 - \phi_1(\theta), \quad \ldots, \quad y_{n-1} = q_{n-1} - \phi_{n-1}(\theta)$$

Step 2: Define a feedback transformation

$$u = (u_1, \ldots, u_{n-1}) \mapsto v = (v_1, \ldots, v_{n-1})$$

such that the dynamics of y become linear

$$\ddot{y} = v, \qquad \ddot{\theta} = N(\theta, \dot{\theta}, y, \dot{y}, \ddot{y})$$

Steps in Computing Transverse Linearization

Given a T-periodic motion $q^\star(t) = \left(q_1^\star(t), q_2^\star(t), \ldots, q_n^\star(t)\right)^T$ of the mechanical system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = R_0(q) + R_2(q, \dot{q}) + B(q)u$$

Step 1: Rewrite the dynamics in the coordinates: θ, y with

$$y_1 = q_1 - \phi_1(\theta), \quad \ldots, \quad y_{n-1} = q_{n-1} - \phi_{n-1}(\theta)$$

Step 2: Define a feedback transformation

$$u = (u_1, \ldots, u_{n-1}) \mapsto v = (v_1, \ldots, v_{n-1})$$

such that the dynamics of y become linear

$$\ddot{y} = v, \qquad \ddot{\theta} = N(\theta, \dot{\theta}, y, \dot{y}, \ddot{y})$$

Steps in Computing Transverse Linearization

Given a T-periodic motion $q^\star(t) = \Big(q_1^\star(t), q_2^\star(t), \ldots, q_n^\star(t)\Big)^T$ of the mechanical system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = R_0(q) + R_2(q, \dot{q}) + B(q)u$$

Step 1: Rewrite the dynamics in the coordinates: θ, y with

$$y_1 = q_1 - \phi_1(\theta), \quad \ldots, \quad y_{n-1} = q_{n-1} - \phi_{n-1}(\theta)$$

Step 2: Define a feedback transformation

$$u = (u_1, \ldots, u_{n-1}) \mapsto v = (v_1, \ldots, v_{n-1})$$

such that the dynamics of y become linear

$$\ddot{y} = v, \qquad \ddot{\theta} = N(\theta, \dot{\theta}, y, \dot{y}, \ddot{y})$$

Steps in Computing Transverse Linearization: θ-dynamics

Since the θ-dynamics with constraints $y = \dot{y} = \ddot{y} = v = 0$

$$\ddot{\theta} = N(\theta, \dot{\theta}, y, \dot{y}, \ddot{y})$$

become

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

Therefore, the dynamics of θ-variable can be rewritten as

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) \;=\; g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\, y + g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\, \dot{y} + g_v(\theta, \dot{\theta}, y, \dot{y})\, \ddot{y}$$

The functions $g_y(\cdot)$, $g_{\dot{y}}(\cdot)$ and $g_v(\cdot)$ are to be found based on

- the Lagrangian $\mathcal{L}(\cdot)$ of the system, the vector functions $R_0(\cdot)$, $R_2(\cdot)$;
- the coupling term $B(\cdot)$;
- the functions $\phi_1(\cdot), \ldots, \phi_{n-1}(\cdot)$ representing the nested parametrization of the motion.

Steps in Computing Transverse Linearization: θ-dynamics

Since the θ-dynamics with constraints $y = \dot{y} = \ddot{y} = v = 0$

$$\ddot{\theta} = N(\theta, \dot{\theta}, y, \dot{y}, \ddot{y})$$

become

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

Therefore, the dynamics of θ-variable can be rewritten as

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) \quad = \quad g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\, y + g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\, \dot{y} + g_v(\theta, \dot{\theta}, y, \dot{y})\, \ddot{y}$$

The functions $g_y(\cdot)$, $g_{\dot{y}}(\cdot)$ and $g_v(\cdot)$ are to be found based on

- the Lagrangian $\mathcal{L}(\cdot)$ of the system, the vector functions $R_0(\cdot)$, $R_2(\cdot)$;
- the coupling term $B(\cdot)$;
- the functions $\phi_1(\cdot), \ldots, \phi_{n-1}(\cdot)$ representing the nested parametrization of the motion.

Computing Transverse Linearization: Passivity Relation for $I(\cdot)$

The dynamics of the controlled mechanical system written in $[\theta,\ y]$-coordinates are

$$\begin{aligned} \alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) &= g_y(\theta,\dot{\theta},\ddot{\theta},y,\dot{y})y + g_{\dot{y}}(\theta,\dot{\theta},\ddot{\theta},y,\dot{y})\dot{y} + g_v(\theta,\dot{\theta},y,\dot{y})v \\ \ddot{y} &= v \end{aligned}$$

We are searching for linearization of its transverse dynamics, i.e. of the variables

$$x_\perp = \Big[\, I(\theta,\dot{\theta},\theta^\star(0),\dot{\theta}^\star(0));\ y;\ \dot{y} \,\Big],$$

in a vicinity of the nominal solution: $\theta(t) = \theta^\star(t)$, $y(t) \equiv 0$.

To do so, one can use the passivity relation for computing $\frac{d}{dt}I(\cdot)$ found previously

$$\begin{aligned} \frac{d}{dt}I(\theta,\dot{\theta},a,b) &\equiv \dot{\theta}\left\{\frac{2}{\alpha(\theta)}W - \frac{2\beta(\theta)}{\alpha(\theta)}I(\theta,\dot{\theta},a,b)\right\} \\ &\equiv \dot{\theta}\left\{\frac{2}{\alpha(\theta)}\Big[g_y(\cdot)y + g_{\dot{y}}(\cdot)\dot{y} + g_v(\cdot)v\Big] - \frac{2\beta(\theta)}{\alpha(\theta)}I(\theta,\dot{\theta},a,b)\right\} \end{aligned}$$

Computing Transverse Linearization: Passivity Relation for $I(\cdot)$

The dynamics of the controlled mechanical system written in $[\theta,\, y]$-coordinates are

$$\begin{aligned} \alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) &= g_y(\theta,\dot{\theta},\ddot{\theta},y,\dot{y})y + g_{\dot{y}}(\theta,\dot{\theta},\ddot{\theta},y,\dot{y})\dot{y} + g_v(\theta,\dot{\theta},y,\dot{y})v \\ \ddot{y} &= v \end{aligned}$$

We are searching for linearization of its transverse dynamics, i.e. of the variables

$$x_\perp = \Big[\, I(\theta,\dot{\theta},\theta^\star(0),\dot{\theta}^\star(0));\; y;\; \dot{y} \,\Big],$$

in a vicinity of the nominal solution: $\theta(t) = \theta^\star(t)$, $y(t) \equiv 0$.

To do so, one can use the passivity relation for computing $\frac{d}{dt}I(\cdot)$ found previously

$$\begin{aligned} \frac{d}{dt}I(\theta,\dot{\theta},a,b) &\equiv \dot{\theta}\left\{\frac{2}{\alpha(\theta)}W - \frac{2\beta(\theta)}{\alpha(\theta)}I(\theta,\dot{\theta},a,b)\right\} \\ &\equiv \dot{\theta}\left\{\frac{2}{\alpha(\theta)}\Big[g_y(\cdot)y + g_{\dot{y}}(\cdot)\dot{y} + g_v(\cdot)v\Big] - \frac{2\beta(\theta)}{\alpha(\theta)}I(\theta,\dot{\theta},a,b)\right\} \end{aligned}$$

Transverse Linearization based on Partial Feedback Linearization

The linear control system with $(2n-1)$ states and $(n-1)$ control inputs

$$\begin{aligned}
\tfrac{d}{dt} I_{\heartsuit}(t) &= \frac{2\,\dot{\theta}^{\star}(t)}{\alpha(\theta^{\star}(t))}\left\{\left[\tilde{g}_y(t) y_{\heartsuit}(t) + \tilde{g}_{\dot{y}}(t)\dot{y}_{\heartsuit}(t) + \tilde{g}_v(t) v_{\heartsuit}(t)\right] - \beta(\theta^{\star}(t)) I_{\heartsuit}(t)\right\} \\
\ddot{y}_{\heartsuit}(t) &= v_{\heartsuit}(t)
\end{aligned}$$

Any **linear** feedback controller of the form

$$v_{\heartsuit}(t) = K(t)\begin{bmatrix} I_{\heartsuit} \\ y_{\heartsuit} \\ \dot{y}_{\heartsuit} \end{bmatrix}, \quad K(t) = K(t+T)$$

that stabilizes the origin of the linear control system can be transformed into a **nonlinear controller that stabilizes orbitally** the nominal periodic motion $q^{\star}(\cdot)$ of the nonlinear underactuated mechanical system

The linear control system with $(2n-1)$ states and $(n-1)$ control inputs

$$\begin{aligned}
\frac{d}{dt} I_{\heartsuit}(t) &= \frac{2\,\dot\theta^\star(t)}{\alpha(\theta^\star(t))}\Big\{\Big[\tilde g_y(t) y_{\heartsuit}(t) + \tilde g_{\dot y}(t)\dot y_{\heartsuit}(t) + \tilde g_v(t) v_{\heartsuit}(t)\Big] - \beta(\theta^\star(t)) I_{\heartsuit}(t)\Big\} \\
\ddot y_{\heartsuit}(t) &= v_{\heartsuit}(t)
\end{aligned}$$

with the functions

$$\begin{aligned}
\tilde g_y(t) &= g_y(\theta,\dot\theta,\ddot\theta,y,\dot y)\Big|_{\substack{\theta=\theta^\star(t),\ \dot\theta=\dot\theta^\star(t) \\ \ddot\theta=\ddot\theta^\star(t),\ y=\dot y=0}} \\
\tilde g_{\dot y}(t) &= g_{\dot y}(\theta,\dot\theta,\ddot\theta,y,\dot y)\Big|_{\substack{\theta=\theta^\star(t),\ \dot\theta=\dot\theta^\star(t) \\ \ddot\theta=\ddot\theta^\star(t),\ y=\dot y=0}} \\
\tilde g_v(t) &= g_v(\theta,\dot\theta,y,\dot y)\Big|_{\substack{\theta=\theta^\star(t),\ \dot\theta=\dot\theta^\star(t) \\ y=\dot y=0}}
\end{aligned}$$

This is the transverse linearization of the underactuated mechanical system in a vicinity of the motion $q^\star(\cdot)$

Any **linear** feedback controller of the form

Transverse Linearization based on Partial Feedback Linearization

The linear control system with $(2n-1)$ states and $(n-1)$ control inputs

$$
\begin{aligned}
\frac{d}{dt} I_{\heartsuit}(t) &= \frac{2\,\dot{\theta}^\star(t)}{\alpha(\theta^\star(t))} \left\{ \left[\tilde{g}_y(t) y_{\heartsuit}(t) + \tilde{g}_{\dot{y}}(t) \dot{y}_{\heartsuit}(t) + \tilde{g}_v(t) v_{\heartsuit}(t) \right] - \beta(\theta^\star(t)) I_{\heartsuit}(t) \right\} \\
\ddot{y}_{\heartsuit}(t) &= v_{\heartsuit}(t)
\end{aligned}
$$

Any **linear** feedback controller of the form

$$
v_{\heartsuit}(t) = K(t) \begin{bmatrix} I_{\heartsuit} \\ y_{\heartsuit} \\ \dot{y}_{\heartsuit} \end{bmatrix}, \quad K(t) = K(t+T)
$$

that stabilizes the origin of the linear control system can be transformed into a **nonlinear controller that stabilizes orbitally** the nominal periodic motion $q^\star(\cdot)$ of the nonlinear underactuated mechanical system

Example: a cart-pendulum system

Example: Planning oscillations of a cart-pendulum system

Some of forced oscillations $[x^*(\cdot), \theta^*(\cdot)]$ of the system

$$\begin{aligned} 2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 &= u \\ \cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta &= 0 \end{aligned}$$

can be written through the nested representation, where

- $\theta^*(\cdot)$ defined as a periodic solution of the reduced dynamics

$$\Big[1 + \cos\theta \cdot \phi'(\theta)\Big]\, \ddot{\theta} + \cos\theta \cdot \phi''(\theta) \cdot \dot{\theta}^2 - g \cdot \sin\theta = 0$$

- $x^*(\cdot)$ defined by $\theta^*(\cdot)$ through the kinematic relation

$$x^*(t) = \phi\left(\theta^*(t)\right)$$

Example: Planning oscillations of a cart-pendulum system

For instance, the system (the dynamics of motion generator θ)

$$\Big[1 + \cos\theta \cdot \phi'(\theta)\Big]\, \ddot{\theta} + \cos\theta \cdot \phi''(\theta)\dot{\theta}^2 - g \cdot \sin\theta = 0$$

has the center at the upright equilibrium $\theta_e = 0$, if the following linear system

$$\ddot{z} + \left[\frac{d}{d\theta}\left(\frac{-g \cdot \sin\theta}{1 + \cos\theta \cdot \phi'(\theta)}\right)\right]\Bigg|_{\theta=0} \cdot z = 0$$

has the center at the origin.

At the same time, the linear system has the form (i.e. a linear oscillator of period $\frac{2\pi}{\omega}$)

$$\ddot{z} + \omega^2 z = 0$$

if the function $\phi(\cdot)$ satisfies the interpolation condition: $\phi'(0) = -\left[1 + \frac{g}{\omega^2}\right]$

Example: Planning oscillations of a cart-pendulum system

For instance, the system (the dynamics of motion generator θ)

$$\Big[1 + \cos\theta \cdot \phi'(\theta)\Big]\,\ddot{\theta} + \cos\theta \cdot \phi''(\theta)\dot{\theta}^2 - g \cdot \sin\theta = 0$$

has the center at the upright equilibrium $\theta_e = 0$, if the following linear system

$$\ddot{z} + \left[\frac{d}{d\theta}\left(\frac{-g \cdot \sin\theta}{1 + \cos\theta \cdot \phi'(\theta)}\right)\right]\Bigg|_{\theta=0} \cdot z = 0$$

has the center at the origin.

At the same time, the linear system has the form (i.e. a linear oscillator of period $\frac{2\pi}{\omega}$)

$$\ddot{z} + \omega^2 z = 0$$

if the function $\phi(\cdot)$ satisfies the interpolation condition: $\phi'(0) = -\left[1 + \dfrac{g}{\omega^2}\right]$

Example: Transverse coordinates for the nominal forced motion

Given a function $\phi(\cdot)$ and $[\theta_0, \dot{\theta}_0]$ such that the solution $\theta(t) = \theta\left(t, \theta_0, \dot{\theta}_0\right)$ of the system

$$\Big[1 + \cos\theta \cdot \phi'(\theta)\Big]\,\ddot{\theta} + \cos\theta \cdot \phi''(\theta)\dot{\theta}^2 - g \cdot \sin\theta = 0$$

creates the nominal T-periodic behavior

$$\theta^*(t) = \theta\left(t, \theta_0, \dot{\theta}_0\right), \quad x^*(t) = \phi\left(\theta\left(t, \theta_0, \dot{\theta}_0\right)\right).$$

The functions of $[\theta, x, \dot{\theta}, \dot{x}]$ that define three transverse coordinates for the motion are

$$x_{1\perp} := y = x - \phi(\theta)$$

$$x_{2\perp} := \dot{y} = \dot{x} - \phi'(\theta)\dot{\theta}$$

$$x_{3\perp} := I = \dot{\theta}^2 - \exp\left\{-2\int_{\theta_0}^{\theta} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\}\left[\left(\dot{\theta}_0\right)^2 - \int_{\theta_0}^{\theta} \exp\left\{-2\int_{\theta_0}^{s} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\} \frac{2\gamma(s)}{\alpha(s)} ds\right]$$

Example: Transverse coordinates for the nominal forced motion

Given a function $\phi(\cdot)$ and $[\theta_0, \dot{\theta}_0]$ such that the solution $\theta(t) = \theta\left(t, \theta_0, \dot{\theta}_0\right)$ of the system

$$\left[1 + \cos\theta \cdot \phi'(\theta)\right]\ddot{\theta} + \cos\theta \cdot \phi''(\theta)\dot{\theta}^2 - g \cdot \sin\theta = 0$$

creates the nominal T-periodic behavior

$$\theta^*(t) = \theta\left(t, \theta_0, \dot{\theta}_0\right), \quad x^*(t) = \phi\left(\theta\left(t, \theta_0, \dot{\theta}_0\right)\right).$$

The functions of $[\theta, x, \dot{\theta}, \dot{x}]$ that define three transverse coordinates for the motion are

$$x_{1\perp} := y = x - \phi(\theta)$$

$$x_{2\perp} := \dot{y} = \dot{x} - \phi'(\theta)\dot{\theta}$$

$$x_{3\perp} := I = \dot{\theta}^2 - \exp\left\{-2\int_{\theta_0}^{\theta} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\}\left[\left(\dot{\theta}_0\right)^2 - \int_{\theta_0}^{\theta} \exp\left\{-2\int_{\theta_0}^{s} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\} \frac{2\gamma(s)}{\alpha(s)} ds\right]$$

Example: Transverse coordinates for the nominal forced motion

Given a function $\phi(\cdot)$ and $[\theta_0, \dot{\theta}_0]$ such that the solution $\theta(t) = \theta\left(t, \theta_0, \dot{\theta}_0\right)$ of the system

$$\left[1 + \cos\theta \cdot \phi'(\theta)\right] \ddot{\theta} + \cos\theta \cdot \phi''(\theta)\dot{\theta}^2 - g \cdot \sin\theta = 0$$

creates the nominal T-periodic behavior

$$\theta^*(t) = \theta\left(t, \theta_0, \dot{\theta}_0\right), \quad x^*(t) = \phi\left(\theta\left(t, \theta_0, \dot{\theta}_0\right)\right).$$

The functions of $[\theta, x, \dot{\theta}, \dot{x}]$ that define three transverse coordinates for the motion are

$$x_{1\perp} := y = x - \phi(\theta)$$

$$x_{2\perp} := \dot{y} = \dot{x} - \phi'(\theta)\dot{\theta}$$

$$x_{3\perp} := I = \dot{\theta}^2 - \exp\left\{-2\int_{\theta_0}^{\theta} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\} \left[\left(\dot{\theta}_0\right)^2 - \int_{\theta_0}^{\theta} \exp\left\{-2\int_{\theta_0}^{s} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\} \frac{2\gamma(s)}{\alpha(s)} ds\right]$$

$$\alpha(s) = 1 + \cos s \cdot \phi'(s), \quad \beta(s) = \cos s \cdot \phi''(s), \quad \gamma(s) = -g \cdot \sin s$$

Example: Transverse coordinates for the nominal forced motion

Given a function $\phi(\cdot)$ and $[\theta_0, \dot\theta_0]$ such that the solution $\theta(t) = \theta\left(t, \theta_0, \dot\theta_0\right)$ of the system

$$\Big[1 + \cos\theta \cdot \phi'(\theta)\Big]\,\ddot\theta + \cos\theta \cdot \phi''(\theta)\dot\theta^2 - g \cdot \sin\theta = 0$$

creates the nominal T-periodic behavior

$$\theta^*(t) = \theta\left(t, \theta_0, \dot\theta_0\right), \quad x^*(t) = \phi\left(\theta\left(t, \theta_0, \dot\theta_0\right)\right).$$

The functions of $[\theta, x, \dot\theta, \dot x]$ that define three transverse coordinates for the motion are

$$x_{1\perp} := y = x - \phi(\theta)$$

$$x_{2\perp} := \dot y = \dot x - \phi'(\theta)\dot\theta$$

$$x_{3\perp} := I = \dot\theta^2 - \exp\left\{-2\int_{\theta_0}^{\theta}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\left[\left(\dot\theta_0\right)^2 - \int_{\theta_0}^{\theta}\exp\left\{-2\int_{\theta_0}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds\right]$$

$$= \dot\theta^2 - h^2\left(\theta, \theta_0, \dot\theta_0\right)$$

Example: Transverse coordinates for the nominal forced motion

Given a function $\phi(\cdot)$ and $[\theta_0, \dot{\theta}_0]$ such that the solution $\theta(t) = \theta\left(t, \theta_0, \dot{\theta}_0\right)$ of the system

$$\left[1 + \cos\theta \cdot \phi'(\theta)\right] \ddot{\theta} + \cos\theta \cdot \phi''(\theta)\dot{\theta}^2 - g \cdot \sin\theta = 0$$

creates the nominal T-periodic behavior

$$\theta^*(t) = \theta\left(t, \theta_0, \dot{\theta}_0\right), \quad x^*(t) = \phi\left(\theta\left(t, \theta_0, \dot{\theta}_0\right)\right).$$

The functions of $[\theta, x, \dot{\theta}, \dot{x}]$ that define three transverse coordinates for the motion are

$$x_{1\perp} := y = x - \phi(\theta)$$

$$x_{2\perp} := \dot{y} = \dot{x} - \phi'(\theta)\dot{\theta}$$

$$x_{3\perp} := I = \dot{\theta}^2 - \exp\left\{-2\int_{\theta_0}^{\theta} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\} \left[\left(\dot{\theta}_0\right)^2 - \int_{\theta_0}^{\theta} \exp\left\{-2\int_{\theta_0}^{s} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\} \frac{2\gamma(s)}{\alpha(s)} ds\right]$$

$$= \dot{\theta}^2 - h^2\left(\theta, \theta_0, \dot{\theta}_0\right) = \left[\dot{\theta} - h\left(\theta, \theta_0, \dot{\theta}_0\right)\right]\left[\dot{\theta} + h\left(\theta, \theta_0, \dot{\theta}_0\right)\right]$$

Example: Transverse coordinates for the nominal forced motion

To verify that the functions $x_{1\perp}(\cdot)$, $x_{2\perp}(\cdot)$ and $x_{3\perp}(\cdot)$ defined as suggested by

$$\begin{aligned}
x_{1\perp} &:= x - \phi(\theta), \qquad x_{2\perp} := \dot{x} - \phi'(\theta)\dot{\theta} \\
x_{3\perp} &:= \dot{\theta}^2 - \exp\left\{-2\int_{\theta_0}^{\theta} \frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\left[\left(\dot{\theta}_0\right)^2 - \int_{\theta_0}^{\theta} \exp\left\{-2\int_{\theta_0}^{s} \frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds\right]
\end{aligned}$$

are independent in a vicinity of the motion, one can check the rank of the Jacobian

$$J(\cdot) = \left.\begin{bmatrix}
\frac{\partial}{\partial x}x_{1\perp} & \frac{\partial}{\partial \dot{x}}x_{1\perp} & \frac{\partial}{\partial \theta}x_{1\perp} & \frac{\partial}{\partial \dot{\theta}}x_{1\perp} \\
\frac{\partial}{\partial x}x_{2\perp} & \frac{\partial}{\partial \dot{x}}x_{2\perp} & \frac{\partial}{\partial \theta}x_{2\perp} & \frac{\partial}{\partial \dot{\theta}}x_{2\perp} \\
\frac{\partial}{\partial x}x_{3\perp} & \frac{\partial}{\partial \dot{x}}x_{3\perp} & \frac{\partial}{\partial \theta}x_{3\perp} & \frac{\partial}{\partial \dot{\theta}}x_{3\perp}
\end{bmatrix}\right|_{x=x^*(t),\, \theta=\theta^*(t),\, \dot{x}=\dot{x}^*(t),\, \dot{\theta}=\dot{\theta}^*(t)}$$

Example: Transverse coordinates for the nominal forced motion

To verify that the functions $x_{1\perp}(\cdot)$, $x_{2\perp}(\cdot)$ and $x_{3\perp}(\cdot)$ defined as suggested by

$$
\begin{aligned}
x_{1\perp} &:= x - \phi(\theta), \qquad x_{2\perp} := \dot{x} - \phi'(\theta)\dot{\theta} \\
x_{3\perp} &:= \dot{\theta}^2 - \exp\left\{-2\int_{\theta_0}^{\theta}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\left[\left(\dot{\theta}_0\right)^2 - \int_{\theta_0}^{\theta}\exp\left\{-2\int_{\theta_0}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds\right]
\end{aligned}
$$

are independent in a vicinity of the motion, one can check the rank of the Jacobian

$$
J(\cdot) = \left.\begin{bmatrix}
\frac{\partial}{\partial x}x_{1\perp} & \frac{\partial}{\partial \dot{x}}x_{1\perp} & \frac{\partial}{\partial \theta}x_{1\perp} & \frac{\partial}{\partial \dot{\theta}}x_{1\perp} \\
\frac{\partial}{\partial x}x_{2\perp} & \frac{\partial}{\partial \dot{x}}x_{2\perp} & \frac{\partial}{\partial \theta}x_{2\perp} & \frac{\partial}{\partial \dot{\theta}}x_{2\perp} \\
\frac{\partial}{\partial x}x_{3\perp} & \frac{\partial}{\partial \dot{x}}x_{3\perp} & \frac{\partial}{\partial \theta}x_{3\perp} & \frac{\partial}{\partial \dot{\theta}}x_{3\perp}
\end{bmatrix}\right|_{x=x^*(t),\, \theta=\theta^*(t),\, \dot{x}=\dot{x}^*(t),\, \dot{\theta}=\dot{\theta}^*(t)}
$$

$$
= \left[\begin{array}{c|c|c|c}
1 & 0 & \phi'(\theta^*(t)) & 0 \\
0 & 1 & \phi''(\theta^*(t))\dot{\theta}^*(t) & \phi'(\theta^*(t)) \\
0 & 0 & \frac{\partial}{\partial \theta}x_{3\perp} & 2\cdot\dot{\theta}^*(t)
\end{array}\right]
$$

Example: Transverse coordinates for the nominal forced motion

To verify that the functions $x_{1\perp}(\cdot)$, $x_{2\perp}(\cdot)$ and $x_{3\perp}(\cdot)$ defined as suggested by

$$x_{1\perp} := x - \phi(\theta), \qquad x_{2\perp} := \dot{x} - \phi'(\theta)\dot{\theta}$$

$$x_{3\perp} := \dot{\theta}^2 - \exp\left\{-2\int_{\theta_0}^{\theta} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\}\left[\left(\dot{\theta}_0\right)^2 - \int_{\theta_0}^{\theta} \exp\left\{-2\int_{\theta_0}^{s} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\} \frac{2\gamma(s)}{\alpha(s)} ds\right]$$

are independent in a vicinity of the motion, one can check the rank of the Jacobian

$$J(\cdot) = \left.\begin{bmatrix} \frac{\partial}{\partial x}x_{1\perp} & \frac{\partial}{\partial \dot{x}}x_{1\perp} & \frac{\partial}{\partial \theta}x_{1\perp} & \frac{\partial}{\partial \dot{\theta}}x_{1\perp} \\ \frac{\partial}{\partial x}x_{2\perp} & \frac{\partial}{\partial \dot{x}}x_{2\perp} & \frac{\partial}{\partial \theta}x_{2\perp} & \frac{\partial}{\partial \dot{\theta}}x_{2\perp} \\ \frac{\partial}{\partial x}x_{3\perp} & \frac{\partial}{\partial \dot{x}}x_{3\perp} & \frac{\partial}{\partial \theta}x_{3\perp} & \frac{\partial}{\partial \dot{\theta}}x_{3\perp} \end{bmatrix}\right|_{x=x^*(t),\, \theta=\theta^*(t),\, \dot{x}=\dot{x}^*(t),\, \dot{\theta}=\dot{\theta}^*(t)}$$

$$= \left[\begin{array}{c|c|c|c} 1 & 0 & \phi'(\theta^*(t)) & 0 \\ 0 & 1 & \phi''(\theta^*(t))\dot{\theta}^*(t) & \phi'(\theta^*(t)) \\ 0 & 0 & \frac{\partial}{\partial \theta}x_{3\perp} & 2\cdot\dot{\theta}^*(t) \end{array}\right] \qquad \left.\frac{\partial}{\partial \theta}x_{3\perp}\right|_* = -2\cdot\ddot{\theta}^*(t)$$

Example: Transverse coordinates for the nominal forced motion

To verify that the functions $x_{1\perp}(\cdot)$, $x_{2\perp}(\cdot)$ and $x_{3\perp}(\cdot)$ defined as suggested by

$$
\begin{aligned}
x_{1\perp} &:= x - \phi(\theta), \qquad x_{2\perp} := \dot{x} - \phi'(\theta)\dot{\theta} \\
x_{3\perp} &:= \dot{\theta}^2 - \exp\left\{-2\int_{\theta_0}^{\theta} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\}\left[\left(\dot{\theta}_0\right)^2 - \int_{\theta_0}^{\theta} \exp\left\{-2\int_{\theta_0}^{s} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\} \frac{2\gamma(s)}{\alpha(s)} ds\right]
\end{aligned}
$$

are independent in a vicinity of the motion, one can check the rank of the Jacobian

$$
\operatorname{rank} J(\cdot) = \operatorname{rank}\left[\begin{array}{c|c|c|c} 1 & 0 & \phi'(\theta^*(t)) & 0 \\ 0 & 1 & \phi''(\theta^*(t))\dot{\theta}^*(t) & \phi'(\theta^*(t)) \\ 0 & 0 & -2\cdot\ddot{\theta}^*(t) & 2\cdot\dot{\theta}^*(t) \end{array}\right] = 2 + \operatorname{rank}\left[-2\cdot\ddot{\theta}^*(t),\ 2\cdot\dot{\theta}^*(t)\right]
$$

Since $\theta^*(t)$ is the cycle of the $(\alpha-\beta-\gamma)$ equation, then there is no a time moment t_c:

$$
\dot{\theta}^*(t_c) = \ddot{\theta}^*(t_c) = 0
$$

$$
\Rightarrow\Rightarrow\Rightarrow \quad \operatorname{rank} J(\cdot)\big|_* = 3 \quad \Leftarrow\Leftarrow\Leftarrow
$$

Example: Transverse coordinates for the nominal forced motion

To verify that the functions $x_{1\perp}(\cdot)$, $x_{2\perp}(\cdot)$ and $x_{3\perp}(\cdot)$ defined as suggested by

$$
\begin{aligned}
x_{1\perp} &:= x - \phi(\theta), \qquad x_{2\perp} := \dot{x} - \phi'(\theta)\dot{\theta} \\
x_{3\perp} &:= \dot{\theta}^2 - \exp\left\{-2\int_{\theta_0}^{\theta} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\} \left[\left(\dot{\theta}_0\right)^2 - \int_{\theta_0}^{\theta} \exp\left\{-2\int_{\theta_0}^{s} \frac{\beta(\tau)}{\alpha(\tau)} d\tau\right\} \frac{2\gamma(s)}{\alpha(s)} ds\right]
\end{aligned}
$$

are independent in a vicinity of the motion, one can check the rank of the Jacobian

$$
\operatorname{rank} J(\cdot) = \operatorname{rank} \left[\begin{array}{c|c|c|c} 1 & 0 & \phi'(\theta^*(t)) & 0 \\ 0 & 1 & \phi''(\theta^*(t))\dot{\theta}^*(t) & \phi'(\theta^*(t)) \\ 0 & 0 & -2\cdot\ddot{\theta}^*(t) & 2\cdot\dot{\theta}^*(t) \end{array}\right] = 2 + \operatorname{rank}\left[-2\cdot\ddot{\theta}^*(t),\ 2\cdot\dot{\theta}^*(t)\right]
$$

Since $\theta^*(t)$ is the cycle of the $(\alpha - \beta - \gamma)$ equation, then there is no a time moment t_c:

$$
\dot{\theta}^*(t_c) = \ddot{\theta}^*(t_c) = 0
$$

$\Rightarrow\Rightarrow\Rightarrow \quad \operatorname{rank} J(\cdot)|_* = 3 \quad \Leftarrow\Leftarrow\Leftarrow$

Example: Transverse coordinates for the nominal forced motion

To verify that the functions $x_{1\perp}(\cdot)$, $x_{2\perp}(\cdot)$ and $x_{3\perp}(\cdot)$ defined as suggested by

$$\begin{aligned} x_{1\perp} &:= x - \phi(\theta), \qquad x_{2\perp} := \dot{x} - \phi'(\theta)\dot{\theta} \\ x_{3\perp} &:= \dot{\theta}^2 - \exp\left\{-2\int_{\theta_0}^{\theta} \frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\left[\left(\dot{\theta}_0\right)^2 - \int_{\theta_0}^{\theta}\exp\left\{-2\int_{\theta_0}^{s}\frac{\beta(\tau)}{\alpha(\tau)}d\tau\right\}\frac{2\gamma(s)}{\alpha(s)}ds\right] \end{aligned}$$

are independent in a vicinity of the motion, one can check the rank of the Jacobian

$$\operatorname{rank} J(\cdot) = \operatorname{rank}\left[\begin{array}{c|c|c|c} 1 & 0 & \phi'(\theta^*(t)) & 0 \\ 0 & 1 & \phi''(\theta^*(t))\dot{\theta}^*(t) & \phi'(\theta^*(t)) \\ 0 & 0 & -2\cdot\ddot{\theta}^*(t) & 2\cdot\dot{\theta}^*(t) \end{array}\right] = 2 + \operatorname{rank}\left[-2\cdot\ddot{\theta}^*(t),\ 2\cdot\dot{\theta}^*(t)\right]$$

Since $\theta^*(t)$ is the cycle of the $(\alpha - \beta - \gamma)$ equation, then there is no a time moment t_c:

$$\dot{\theta}^*(t_c) = \ddot{\theta}^*(t_c) = 0$$

$$\Rightarrow\Rightarrow\Rightarrow \quad \operatorname{rank} J(\cdot)\big|_{*} = 3 \quad \Leftarrow\Leftarrow\Leftarrow$$

Example: Linearization of transverse coordinates for a forced motion

To derive the linearization of transverse dynamics, the equations of motion of the system

$$2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 = u, \qquad \cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta = 0$$

should be re-written in new variables $[\theta, y]$ and with new control input v

$$\begin{aligned} \alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) &= g_y(\theta,\dot{\theta},\ddot{\theta},y,\dot{y})y + g_{\dot{y}}(\theta,\dot{\theta},\ddot{\theta},y,\dot{y})\dot{y} + g_v(\theta,\dot{\theta},y,\dot{y})v \\ \ddot{y} &= v \end{aligned}$$

The task is then

Step 1: to define the feedback transformation: $u \mapsto v$

If done, the linearization of the transverse coordinates can be written symbolically as:

$$\dot{I}_\heartsuit = a_{11}(t)I_\heartsuit + a_{12}(t)y_\heartsuit + a_{13}(t)\dot{y}_\heartsuit + b_1(t)v_\heartsuit, \qquad \ddot{y}_\heartsuit = v_\heartsuit$$

Example: Linearization of transverse coordinates for a forced motion

To derive the linearization of transverse dynamics, the equations of motion of the system

$$2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 = u, \qquad \cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta = 0$$

should be re-written in new variables $[\theta, y]$ and with new control input v

$$\begin{aligned} \alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) &= g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})y + g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\dot{y} + g_v(\theta, \dot{\theta}, y, \dot{y})v \\ \ddot{y} &= v \end{aligned}$$

The task is then

Step 1: to define the feedback transformation: $u \mapsto v$

If done, the linearization of the transverse coordinates can be written symbolically as:

$$\dot{I}_\heartsuit = a_{11}(t)I_\heartsuit + a_{12}(t)y_\heartsuit + a_{13}(t)\dot{y}_\heartsuit + b_1(t)v_\heartsuit, \qquad \ddot{y}_\heartsuit = v_\heartsuit$$

Example: Linearization of transverse coordinates for a forced motion

To derive the linearization of transverse dynamics, the equations of motion of the system

$$2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 = u, \qquad \cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta = 0$$

should be re-written in new variables $[\theta, y]$ and with new control input v

$$\begin{aligned} \alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) &= g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})y + g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\dot{y} + g_v(\theta, \dot{\theta}, y, \dot{y})v \\ \ddot{y} &= v \end{aligned}$$

The task is then

Step 1: to define the feedback transformation: $u \mapsto v$

Step 2: to determine the functions $g_y(\cdot)$, $g_{\dot{y}}(\cdot)$, $g_v(\cdot)$

If done, the linearization of the transverse coordinates can be written symbolically as:

$$\dot{I}_\heartsuit = a_{11}(t)I_\heartsuit + a_{12}(t)y_\heartsuit + a_{13}(t)\dot{y}_\heartsuit + b_1(t)v_\heartsuit, \qquad \ddot{y}_\heartsuit = v_\heartsuit$$

Example: Linearization of transverse coordinates for a forced motion

To derive the linearization of transverse dynamics, the equations of motion of the system

$$2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 = u, \qquad \cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta = 0$$

should be re-written in new variables $[\theta, y]$ and with new control input v

$$\begin{aligned} \alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) &= g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})y + g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\dot{y} + g_v(\theta, \dot{\theta}, y, \dot{y})v \\ \ddot{y} &= v \end{aligned}$$

The task is then

Step 1: to define the feedback transformation: $u \mapsto v$

Step 2: to determine the functions $g_y(\cdot)$, $g_{\dot{y}}(\cdot)$, $g_v(\cdot)$

If done, the linearization of the transverse coordinates can be written symbolically as:

$$\dot{I}_{\heartsuit} = a_{11}(t)I_{\heartsuit} + a_{12}(t)y_{\heartsuit} + a_{13}(t)\dot{y}_{\heartsuit} + b_1(t)v_{\heartsuit}, \qquad \ddot{y}_{\heartsuit} = v_{\heartsuit}$$

Example: Linearization of transverse coordinates for a forced motion. Step 1

To define the feedback transform $u \mapsto v$ that converts the y-dynamics into $\ddot{y} = v$, one should

- Derive the equation $\ddot{y} = A(q, \dot{q}) + B(q, \dot{q})u$

Given $y := x - \phi(\theta)$, to determine the functions $A(\cdot)$ and $B(\cdot)$ one can use the relation

$$\ddot{y} = \ddot{x} - \phi''(\theta) \cdot \dot{\theta}^2 - \phi'(\theta) \cdot \ddot{\theta}$$

Homework: Complete the calculations and find the expressions for the functions $A(\cdot)$ and $B(\cdot)$

Example: Linearization of transverse coordinates for a forced motion. Step 1

To define the feedback transform $u \mapsto v$ that converts the y-dynamics into $\ddot{y} = v$, one should

- Derive the equation $\ddot{y} = A(q, \dot{q}) + B(q, \dot{q})u$
- Solve the equation $A(q, \dot{q}) + B(q, \dot{q})u = v$ with respect to u, i.e. $u = \dfrac{v - A(q, \dot{q})}{B(q, \dot{q})}$

Given $y := x - \phi(\theta)$, to determine the functions $A(\cdot)$ and $B(\cdot)$ one can use the relation

$$\ddot{y} = \ddot{x} - \phi''(\theta) \cdot \dot{\theta}^2 - \phi'(\theta) \cdot \ddot{\theta}$$

Homework: Complete the calculations and find the expressions for the functions $A(\cdot)$ and $B(\cdot)$

Example: Linearization of transverse coordinates for a forced motion. Step 1

To define the feedback transform $u \mapsto v$ that converts the y-dynamics into $\ddot{y} = v$, one should

- Derive the equation $\ddot{y} = A(q, \dot{q}) + B(q, \dot{q})u$
- Solve the equation $A(q, \dot{q}) + B(q, \dot{q})u = v$ with respect to u, i.e. $u = \dfrac{v - A(q, \dot{q})}{B(q, \dot{q})}$

  Caution: The answer is often local, it exists only if $B(\cdot)$ can be inverted on the motion!

Given $y := x - \phi(\theta)$, to determine the functions $A(\cdot)$ and $B(\cdot)$ one can use the relation

$$\ddot{y} = \ddot{x} - \phi''(\theta) \cdot \dot{\theta}^2 - \phi'(\theta) \cdot \ddot{\theta}$$

Homework: Complete the calculations and find the expressions for the functions $A(\cdot)$ and $B(\cdot)$

Example: Linearization of transverse coordinates for a forced motion. Step 1

To define the feedback transform $u \mapsto v$ that converts the y-dynamics into $\ddot{y} = v$, one should

- Derive the equation $\ddot{y} = A(q, \dot{q}) + B(q, \dot{q})u$
- Solve the equation $A(q, \dot{q}) + B(q, \dot{q})u = v$ with respect to u, i.e. $u = \dfrac{v - A(q, \dot{q})}{B(q, \dot{q})}$

  Caution: The answer is often local, it exists only if $B(\cdot)$ can be inverted on the motion!

Given $y := x - \phi(\theta)$, to determine the functions $A(\cdot)$ and $B(\cdot)$ one can use the relation

$$\ddot{y} = \ddot{x} - \phi''(\theta) \cdot \dot{\theta}^2 - \phi'(\theta) \cdot \ddot{\theta}$$

Homework: Complete the calculations and find the expressions for the functions $A(\cdot)$ and $B(\cdot)$

Example: Linearization of transverse coordinates for a forced motion. Step 1

To define the feedback transform $u \mapsto v$ that converts the y-dynamics into $\ddot{y} = v$, one should

- Derive the equation $\ddot{y} = A(q, \dot{q}) + B(q, \dot{q})u$
- Solve the equation $A(q, \dot{q}) + B(q, \dot{q})u = v$ with respect to u, i.e. $u = \dfrac{v - A(q, \dot{q})}{B(q, \dot{q})}$

  Caution: The answer is often local, it exists only if $B(\cdot)$ can be inverted on the motion!

Given $y := x - \phi(\theta)$, to determine the functions $A(\cdot)$ and $B(\cdot)$ one can use the relation

$$\ddot{y} = \ddot{x} - \phi''(\theta) \cdot \dot{\theta}^2 - \phi'(\theta) \cdot \ddot{\theta}$$

and the dynamics of the system

$$\begin{bmatrix} 2 & \cos\theta \\ \cos\theta & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \sin\theta \cdot \dot{\theta}^2 \\ g \cdot \sin\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Homework: Complete the calculations and find the expressions for the functions $A(\cdot)$ and $B(\cdot)$

Example: Linearization of transverse coordinates for a forced motion. Step 1

To define the feedback transform $u \mapsto v$ that converts the y-dynamics into $\ddot{y} = v$, one should

- Derive the equation $\ddot{y} = A(q, \dot{q}) + B(q, \dot{q})u$
- Solve the equation $A(q, \dot{q}) + B(q, \dot{q})u = v$ with respect to u, i.e. $u = \dfrac{v - A(q, \dot{q})}{B(q, \dot{q})}$

  Caution: The answer is often local, it exists only if $B(\cdot)$ can be inverted on the motion!

Given $y := x - \phi(\theta)$, to determine the functions $A(\cdot)$ and $B(\cdot)$ one can use the relation

$$\ddot{y} = \ddot{x} - \phi''(\theta) \cdot \dot{\theta}^2 - \phi'(\theta) \cdot \ddot{\theta}$$

and the dynamics of the system

$$\begin{bmatrix} 2 & \cos\theta \\ \cos\theta & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \sin\theta \cdot \dot{\theta}^2 \\ g \cdot \sin\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad \Rightarrow \quad \begin{cases} \ddot{x} &= f_1(\cdot) + g_1(\cdot)u \\ \ddot{\theta} &= f_2(\cdot) + g_2(\cdot)u \end{cases}$$

Homework: Complete the calculations and find the expressions for the functions $A(\cdot)$ and $B(\cdot)$

Example: Linearization of transverse coordinates for a forced motion. Step 1

To define the feedback transform $u \mapsto v$ that converts the y-dynamics into $\ddot{y} = v$, one should

- Derive the equation $\ddot{y} = A(q,\dot{q}) + B(q,\dot{q})u$
- Solve the equation $A(q,\dot{q}) + B(q,\dot{q})u = v$ with respect to u, i.e. $u = \dfrac{v - A(q,\dot{q})}{B(q,\dot{q})}$

  Caution: The answer is often local, it exists only if $B(\cdot)$ can be inverted on the motion!

Given $y := x - \phi(\theta)$, to determine the functions $A(\cdot)$ and $B(\cdot)$ one can use the relation

$$\ddot{y} = \ddot{x} - \phi''(\theta)\cdot\dot{\theta}^2 - \phi'(\theta)\cdot\ddot{\theta}$$

and the dynamics of the system

$$\begin{bmatrix} 2 & \cos\theta \\ \cos\theta & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \sin\theta\cdot\dot{\theta}^2 \\ g\cdot\sin\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad \Rightarrow \quad \left\{ \begin{array}{lcl} \ddot{x} & = & f_1(\cdot) + g_1(\cdot)u \\ \ddot{\theta} & = & f_2(\cdot) + g_2(\cdot)u \end{array} \right.$$

Homework: Complete the calculations and find the expressions for the functions $A(\cdot)$ and $B(\cdot)$

Example: Linearization of transverse coordinates for a forced motion. Step 2

To obtain the functions $g_y(\cdot)$, $g_{\dot y}(\cdot)$, $g_v(\cdot)$ for the system

$$\begin{aligned} \alpha(\theta)\ddot\theta + \beta(\theta)\dot\theta^2 + \gamma(\theta) &= g_y(\theta,\dot\theta,\ddot\theta,y,\dot y)y + g_{\dot y}(\theta,\dot\theta,\ddot\theta,y,\dot y)\dot y + g_v(\theta,\dot\theta,y,\dot y){\color{red}v} \\ \ddot y &= {\color{red}v} \end{aligned}$$

which represents the equations of motion of the system

$$2\cdot\ddot x + \cos\theta\cdot\ddot\theta - \sin\theta\cdot\dot\theta^2 = {\color{red}u}, \qquad \cos\theta\cdot\ddot x + \ddot\theta - g\cdot\sin\theta = 0$$

re-written in variables $[\theta, y]$ with control input ${\color{red}v}$, consider its 2nd equation (passive dynamics)

Given the definition of $y := x - \phi(\theta)$, ona can determine $\ddot x$ as a function of new variables from

$$\ddot y = \ddot x - \phi''(\theta)\cdot\dot\theta^2 - \phi'(\theta)\cdot\ddot\theta$$

Therefore, the second equation of the original dynamics can be re-written as

$$\Big[1+\cos\theta\cdot\phi'(\theta)\Big]\cdot\ddot\theta + \cos\theta\cdot\phi''(\theta)\cdot\dot\theta^2 - g\cdot\sin\theta = \underbrace{\mathbf{0}}_{g_y(\cdot)}\cdot y + \underbrace{\mathbf{0}}_{g_{\dot y}(\cdot)}\cdot\dot y + \underbrace{-\cos\theta}_{g_v(\cdot)}\cdot\ddot y$$

Example: Linearization of transverse coordinates for a forced motion. Step 2

To obtain the functions $g_y(\cdot)$, $g_{\dot y}(\cdot)$, $g_v(\cdot)$ for the system

$$\begin{aligned}\alpha(\theta)\ddot\theta + \beta(\theta)\dot\theta^2 + \gamma(\theta) &= g_y(\theta,\dot\theta,\ddot\theta,y,\dot y)y + g_{\dot y}(\theta,\dot\theta,\ddot\theta,y,\dot y)\dot y + g_v(\theta,\dot\theta,y,\dot y){\color{red}v}\\ \ddot y &= {\color{red}v}\end{aligned}$$

which represents the equations of motion of the system

$$2\cdot\ddot x + \cos\theta\cdot\ddot\theta - \sin\theta\cdot\dot\theta^2 = {\color{red}u}, \qquad \cos\theta\cdot\ddot x + \ddot\theta - g\cdot\sin\theta = 0$$

re-written in variables $[\theta, y]$ with control input ${\color{red}v}$, consider its 2nd equation (passive dynamics)

Given the definition of $y := x - \phi(\theta)$, ona can determine $\ddot x$ as a function of new variables from

$$\ddot y = \ddot x - \phi''(\theta)\cdot\dot\theta^2 - \phi'(\theta)\cdot\ddot\theta$$

Therefore, the second equation of the original dynamics can be re-written as

$$\Big[1+\cos\theta\cdot\phi'(\theta)\Big]\cdot\ddot\theta + \cos\theta\cdot\phi''(\theta)\cdot\dot\theta^2 - g\cdot\sin\theta = \underbrace{\mathbf{0}}_{g_y(\cdot)}\cdot y + \underbrace{\mathbf{0}}_{g_{\dot y}(\cdot)}\cdot\dot y + \underbrace{-\cos\theta}_{g_v(\cdot)}\cdot\ddot y$$

Example: Linearization of transverse coordinates for a forced motion. Step 2

To obtain the functions $g_y(\cdot)$, $g_{\dot{y}}(\cdot)$, $g_v(\cdot)$ for the system

$$\begin{aligned} \alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) &= g_y(\theta,\dot{\theta},\ddot{\theta},y,\dot{y})y + g_{\dot{y}}(\theta,\dot{\theta},\ddot{\theta},y,\dot{y})\dot{y} + g_v(\theta,\dot{\theta},y,\dot{y})v \\ \ddot{y} &= v \end{aligned}$$

which represents the equations of motion of the system

$$2\cdot\ddot{x} + \cos\theta\cdot\ddot{\theta} - \sin\theta\cdot\dot{\theta}^2 = u, \qquad \cos\theta\cdot\ddot{x} + \ddot{\theta} - g\cdot\sin\theta = 0$$

re-written in variables $[\theta, y]$ with control input v, consider its 2nd equation (passive dynamics)

Given the definition of $y := x - \phi(\theta)$, ona can determine $\ddot{x}$ as a function of new variables from

$$\ddot{y} = \ddot{x} - \phi''(\theta)\cdot\dot{\theta}^2 - \phi'(\theta)\cdot\ddot{\theta}$$

Therefore, the second equation of the original dynamics can be re-written as

$$\cos\theta\cdot\left[\ddot{y} + \phi''(\theta)\cdot\dot{\theta}^2 + \phi'(\theta)\cdot\ddot{\theta}\right] + \ddot{\theta} - g\cdot\sin\theta = 0$$

Therefore, the second equation of the original dynamics can be re-written as

Example: Linearization of transverse coordinates for a forced motion. Step 2

To obtain the functions $g_y(\cdot)$, $g_{\dot y}(\cdot)$, $g_v(\cdot)$ for the system

$$\begin{aligned} \alpha(\theta)\ddot\theta + \beta(\theta)\dot\theta^2 + \gamma(\theta) &= g_y(\theta,\dot\theta,\ddot\theta,y,\dot y)y + g_{\dot y}(\theta,\dot\theta,\ddot\theta,y,\dot y)\dot y + g_v(\theta,\dot\theta,y,\dot y)v \\ \ddot y &= v \end{aligned}$$

which represents the equations of motion of the system

$$2\cdot\ddot x + \cos\theta\cdot\ddot\theta - \sin\theta\cdot\dot\theta^2 = u, \qquad \cos\theta\cdot\ddot x + \ddot\theta - g\cdot\sin\theta = 0$$

re-written in variables $[\theta, y]$ with control input v, consider its 2nd equation (passive dynamics)

Given the definition of $y := x - \phi(\theta)$, ona can determine $\ddot x$ as a function of new variables from

$$\ddot y = \ddot x - \phi''(\theta)\cdot\dot\theta^2 - \phi'(\theta)\cdot\ddot\theta$$

Therefore, the second equation of the original dynamics can be re-written as

$$\Big[1 + \cos\theta\cdot\phi'(\theta)\Big]\cdot\ddot\theta + \cos\theta\cdot\phi''(\theta)\cdot\dot\theta^2 - g\cdot\sin\theta = -\cos\theta\cdot\ddot y$$

Therefore, the second equation of the original dynamics can be re-written as

Example: Linearization of transverse coordinates for a forced motion. Step 2

To obtain the functions $g_y(\cdot)$, $g_{\dot{y}}(\cdot)$, $g_v(\cdot)$ for the system

$$\begin{aligned} \alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) &= g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})y + g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})\dot{y} + g_v(\theta, \dot{\theta}, y, \dot{y})v \\ \ddot{y} &= v \end{aligned}$$

which represents the equations of motion of the system

$$2 \cdot \ddot{x} + \cos\theta \cdot \ddot{\theta} - \sin\theta \cdot \dot{\theta}^2 = u, \qquad \cos\theta \cdot \ddot{x} + \ddot{\theta} - g \cdot \sin\theta = 0$$

re-written in variables $[\theta, y]$ with control input v, consider its 2nd equation (passive dynamics)

Given the definition of $y := x - \phi(\theta)$, ona can determine $\ddot{x}$ as a function of new variables from

$$\ddot{y} = \ddot{x} - \phi''(\theta) \cdot \dot{\theta}^2 - \phi'(\theta) \cdot \ddot{\theta}$$

Therefore, the second equation of the original dynamics can be re-written as

$$\Big[1 + \cos\theta \cdot \phi'(\theta)\Big] \cdot \ddot{\theta} + \cos\theta \cdot \phi''(\theta) \cdot \dot{\theta}^2 - g \cdot \sin\theta = \underbrace{\mathbf{0}}_{g_y(\cdot)} \cdot y + \underbrace{\mathbf{0}}_{g_{\dot{y}}(\cdot)} \cdot \dot{y} + \underbrace{-\cos\theta}_{g_v(\cdot)} \cdot \ddot{y}$$

Example: Transverse linearization. Final form

For the example

$$\alpha(\theta) = 1 + \cos\theta \cdot \phi'(\theta), \quad \beta(\theta) = \cos\theta \cdot \phi''(\theta), \quad g_y(\cdot) = g_{\dot{y}}(\cdot) \equiv 0, \quad g_v(\cdot) = -\cos\theta$$

Therefore, the linear control system with three states and one control input

$$\begin{aligned} \tfrac{d}{dt} I_\heartsuit(t) &= \frac{2\,\dot{\theta}^\star(t)}{\alpha(\theta^\star(t))}\left\{\left[\tilde{g}_y(t) y_\heartsuit(t) + \tilde{g}_{\dot{y}}(t)\dot{y}_\heartsuit(t) + \tilde{g}_v(t) v_\heartsuit(t)\right] - \beta(\theta^\star(t)) I_\heartsuit(t)\right\} \\ \ddot{y}_\heartsuit(t) &= v_\heartsuit(t) \end{aligned}$$

with the functions

$$\begin{aligned} \tilde{g}_y(t) &= g_y(\theta,\dot{\theta},\ddot{\theta},y,\dot{y})\Big|_{\substack{\theta=\theta^\star(t),\ \dot{\theta}=\dot{\theta}^\star(t) \\ \ddot{\theta}=\ddot{\theta}^\star(t),\ y=\dot{y}=0}} \\ \tilde{g}_{\dot{y}}(t) &= g_{\dot{y}}(\theta,\dot{\theta},\ddot{\theta},y,\dot{y})\Big|_{\substack{\theta=\theta^\star(t),\ \dot{\theta}=\dot{\theta}^\star(t) \\ \ddot{\theta}=\ddot{\theta}^\star(t),\ y=\dot{y}=0}} \\ \tilde{g}_v(t) &= g_v(\theta,\dot{\theta},y,\dot{y})\Big|_{\substack{\theta=\theta^\star(t),\ \dot{\theta}=\dot{\theta}^\star(t) \\ y=\dot{y}=0}} \end{aligned}$$

takes the form: $\dot{I}_\heartsuit = a_{11}(t) I_\heartsuit + a_{12}(t) y_\heartsuit + a_{13}(t)\dot{y}_\heartsuit + b_1(t) v_\heartsuit, \quad \ddot{y}_\heartsuit = v_\heartsuit$

Example: Transverse linearization. Final form

For the example

$$\alpha(\theta) = 1 + \cos\theta \cdot \phi'(\theta), \quad \beta(\theta) = \cos\theta \cdot \phi''(\theta), \quad g_y(\cdot) = g_{\dot y}(\cdot) \equiv 0, \quad g_v(\cdot) = -\cos\theta$$

Therefore, the linear control system with three states and one control input

$$\begin{aligned} \tfrac{d}{dt} I_\heartsuit(t) &= \frac{2\,\dot\theta^\star(t)}{\alpha(\theta^\star(t))}\left\{\left[\tilde g_y(t) y_\heartsuit(t) + \tilde g_{\dot y}(t)\dot y_\heartsuit(t) + \tilde g_v(t) v_\heartsuit(t)\right] - \beta(\theta^\star(t)) I_\heartsuit(t)\right\} \\ \ddot y_\heartsuit(t) &= v_\heartsuit(t) \end{aligned}$$

takes the form: $\dot I_\heartsuit = a_{11}(t) I_\heartsuit + a_{12}(t) y_\heartsuit + a_{13}(t)\dot y_\heartsuit + b_1(t) v_\heartsuit, \quad \ddot y_\heartsuit = v_\heartsuit$

with $a_{12}(t) = a_{13}(t) \equiv 0$, $\forall\, t$ and

$$a_{11}(t) = -\frac{2\,\dot\theta^\star(t)\cdot\beta(\theta^\star(t))}{\alpha(\theta^\star(t))} = -\frac{2\,\dot\theta^\star(t)\cdot\cos\theta^\star(t)\cdot\phi''(\theta^\star(t))}{1+\cos\theta^\star(t)\cdot\phi'(\theta^\star(t))}$$

$$b_1(t) = -\frac{2\,\dot\theta^\star(t)\cdot\cos(\theta^\star(t))}{\alpha(\theta^\star(t))} = -\frac{2\,\dot\theta^\star(t)\cdot\cos(\theta^\star(t))}{1+\cos\theta^\star(t)\cdot\phi'(\theta^\star(t))}$$